Well-Posedness for a Class of Fourth Order Diffusions for Image Processing

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Abstract. A number of image denoising models based on higher order parabolic partial differential equations (PDEs) have been proposed in an effort to overcome some of the problems attendant to second order methods such as the famous Perona-Malik model. However, there is little analysis of these equations to be found in the literature. In this paper, methods of maximal regularity are used to prove the existence of unique local solutions to a class of fourth order PDEs for noise removal. The proof is laid out explicitly for two newly proposed fourth order models, and an outline is given for how to apply the techniques to other proposed models.

1. Introduction

1.1. Second Order Diffusions and Regularizations

Use of nonlinear PDEs for image denoising dates back to 1990 when Perona and Malik [30] proposed a dramatic change in the scale space diffusion method standard for smoothing noisy images. The scale space method, introduced by Witkin in 1983 [34], smoothes an image by convolving it with Gaussian kernels on a scale of variances. Equivalently, the smoothed images may be viewed as solution of the linear heat equation

$$u_t = \gamma \Delta u, \quad u(0) = u_0 \text{ (original image)},$$

where the diffusion coefficient γ is constant. While this is an effective denoising method, it is ignorant of features in the image that one may wish to preserve; that is, it cannot distinguish noise from edges, resulting in the whole image becoming blurred. This method then needs to be complemented by a second processing step to locate and reintroduce edges. Perona and Malik's idea was to remove noise and preserve edges in a single step by replacing γ with a nonlinear diffusivity which would inhibit diffusion across edges. This can be accomplished by observing that edges in an image correspond to regions of high gradient. Thus Perona and Malik consider $\gamma = g(|\nabla u|^2)$, where

$g(\cdot)$ is chosen appropriately so as to slow diffusion (become small) when the edge detector $|\nabla u|^2$ is large. One possibility for such a function suggested by Perona and Malik is:

$$g(s^2) = \frac{1}{1+c^2s^2}, \qquad \text{for a constant } c > 0. \tag{1}$$

In their paper, Perona and Malik propose discrete equations which can be interpreted as discretizations of the second order parabolic PDE

$$u_t - \nabla \cdot \big(g(|\nabla u|^2)\nabla u\big) = 0. \tag{2}$$

Such discretizations are effective at removing noise and preserving edges, but they exhibit a couple of notable shortcomings. They can create false edges, a phenomenon known as staircasing, as well as a blocky, cartoonish appearance in the smoothed image. Computing the divergence in (2) with the choice of g given above (with $c = 1$) yields

$$\begin{aligned}\nabla \cdot \Big(\frac{1}{1+|\nabla u|^2}\nabla u\Big) &= \frac{1}{1+|\nabla u|^2}\Big[\Delta u - 2\frac{\nabla u^T D^2 u \nabla u}{1+|\nabla u|^2}\Big] \\ &= \frac{1}{1+|\nabla u|^2}\Big[\partial_\tau^2 u + \Big(1 - 2\frac{|\nabla u|^2}{1+|\nabla u|^2}\Big)\partial_\nu^2 u\Big],\end{aligned}$$

where $\nu = \nabla u/|\nabla u|$ and τ is a unit vector orthogonal to ν. Thus diffusion in the direction of ν may change sign in regions of steep gradient, resulting in backward diffusion, which is known to cause unstable behavior. This unstable behavior manifests itself as staircasing in numerical implementations. The forward-backward nature of (2) is exploited in [25] to prove its ill-posedness.

Perona and Malik suggested the diffusivity (1) in [30] with the idea that the backward diffusion across gradients would cause the sharpening of edges. (An alternate diffusivity with similar behavior, $g(s^2) = e^{-c^2s^2}$, is also suggested.) Indeed, the best feature of the Perona-Malik model, its edge detection capability, is the source of its ill-posedness. Many authors have searched for regularizations of (2) capable of preventing backward diffusion, thus leading to well-posedness, while still preserving edges. Charbonnier, et. al., propose in [11] the less degenerate diffusivity function,

$$g(s^2) = \frac{1}{\sqrt{1+c^2s^2}},$$

which does not lead to backward diffusion. See [15] for a more thorough discussion of different choices for diffusivities.

Others have suggested a regularization of the edge detector $|\nabla u|$, the argument inside the diffusivity function. Particularly novel and effective regularizations involving fractional derivatives are proposed by Guidotti and Lambers in [18–20]. Their proposed models are based on the equations

$$u_t - \nabla \cdot \big(g(|\nabla^{1-\varepsilon} u|^2)\nabla u\big) = 0 \tag{3}$$

and

$$u_t - g\big([(-\Delta)^{1-\varepsilon}]^2\big)\Delta u = 0, \tag{4}$$

coupled with appropriate initial and boundary conditions, where $\varepsilon \in (0,1)$. Fractional derivatives have been shown to be an effective tool for both regularization [14] and edge detection [29]. Indeed, it is shown in [18, 19] that (3) and (4) are locally well-posed and that (3) admits characteristic functions of smooth sets as stationary solutions. Additionally, numerical experiments suggest that these two models produce significantly less staircasing than does (2) or other second order diffusion equations.

The use of fractional derivatives as an edge detection tools has proven more robust in the presence of noise (cf. [27, 29]). This is most likely due to the fact that the corresponding kernels, while still singular, are non-local and thus provide some degree of averaging. The use of the Laplacian or its fractional powers as an edge detector is sensible for very sharp edges when the maximal curvature is observed very close to the edge itself while would become more debatable for less pronounced edges where maximal curvature would be dislocated as compared to the site the edge.

1.2. Fourth Order Diffusions

Another line of research applies the ideas of Perona and Malik to higher order equations in an effort to reduce staircasing and cartoonish effects. Equation (2) can be shown to be associated with the minimization of a first order energy functional. Many authors have considered instead second order energy functionals. Examples of such higher order variational methods can be found in [5, 9, 10, 26, 28]. A fourth order PDE associated with a second order energy functional is that proposed by You and Kaveh [35],

$$u_t + \Delta\Big(g\big((\Delta u)^2\big)\Delta u\Big) = 0. \tag{5}$$

Tumblin and Turk consider in [32] a fourth order PDE model designed to simplify an image based on its curvature,

$$u_t + \nabla\cdot\Big(g\big(\frac{1}{2}(u_{xx}^2 + u_{yy}^2) + u_{xy}^2\big)\nabla\Delta u\Big) = 0. \tag{6}$$

Bertozzi and Greer propose in [6] a modification of (6),

$$u_t + \nabla\cdot\Big(g\big((\Delta u)^2\big)\nabla\Delta u\Big) = 0. \tag{7}$$

A high order generalization of Perona-Malik considered by Wei in [33] reads

$$u_t + \nabla\cdot\Big(g\big(|\nabla u|^2\big)\nabla\Delta u\Big) = 0. \tag{8}$$

Hajiaboli proposes in [22] a modification of (5) using $|\nabla u|$ as an edge detector, similar to

$$u_t + \Delta\Big(g\big(|\nabla u|^2\big)\Delta u\Big) = 0, \tag{9}$$

but with also a modified diffusivity function g. The same author considers another modification of (5) in [23] which utilizes an anisotropic diffusion tensor.

The authors propose in [21] two modifications of (5) in which the edge detector is replaced with the fractional gradient used in (3) and the fractional Laplacian operator used in (4), respectively. These two new equations read

$$u_t + \Delta\Big(g\big(|\nabla^{1-\varepsilon}u|\big)\Delta u\Big) = 0 \tag{10}$$

and

$$u_t + \nabla\cdot\Big(g\big((-\Delta)^{1-\varepsilon}u\big)\nabla\Delta\Big) = 0. \tag{11}$$

A widely used heuristic argument for the use of fourth order models is that they would avoid staircasing by virtue of having "affine functions" as stationary solutions (in contrast to some second order models where piecewise constant functions can be shown to be equilibria [18]). A rigorous analysis of the structure of the set of equilibria for fourth models is an interesting open question that is not addressed in this paper.

Finally we refer to [7] for a survey of other denoising methods, including those not based on PDEs.

1.3. Well-Posedness

Denoising experiments performed with some of the above equations (see, in particular, [15, 21–23, 28, 35]) show that fourth order models can avoid the staircasing and cartoonish effects of second order models while still removing noise and preserving edges. However, little analysis has been conducted on these fourth order equations.

In [17], Greer and Bertozzi prove the well-posedness of regularizations of (6) and (5). This result is in the same spirit as the many results regarding well-posedness of regularizations of the Perona-Malik equation, cf. [3, 8, 18], for example.

Bertozzi and Greer address in [6] the existence of solutions to a special case of (7). Their work relies on the structure of the particular choice of diffusivity function g, and is therefore difficult to generalize.

Didas, Weickert, and Burgeth, in [15], show that higher order generalizations of (2) such as (5) are L_2-stable and preserve average grey value, as well as higher order moments. The proofs in [15] can be easily modified to apply to (10) and (11) with appropriate boundary conditions. Since there is generally no maximum principle for fourth (or higher) order diffusions, it is not expected, or at least not obvious, that fourth order equations would be L_∞-stable.

The effects of different choices of diffusivity function on (5) are also discussed in [15]. It is observed that the Perona-Malik diffusivity (1) promotes forward-backward diffusion, and consequently the sharpening of edges, in (5) as it does in (2). Indeed, expanding out the Laplacian term in (5) with $g(s^2) = (1+s^2)^{-1}$ reveals that

$$\Delta\Big(\frac{1}{1+(\Delta u)^2}\Delta u\Big) = \frac{1}{1+(\Delta u)^2}\Big(1 - \frac{2(\Delta u)^2}{1+(\Delta u)^2}\Big)\Delta^2 u \\ + [\text{lower order terms}],$$

and so backward diffusion can occur when $|\Delta u|$ is large. This likely implies that (5), like (2), is ill-posed. Unlike in (2), this backward diffusion is not necessarily in a direction perpendicular to edges, so in the fourth order case it is not obvious that backward diffusion contributes to edge sharpening as it does in the second order case.

Replacing the edge detector $|\Delta u|$ in (5) with $|\nabla u|$, as used in (2), eliminates this backward behavior. Expanding the Laplacian term of (9) in one dimension shows

$$\begin{aligned}\big(g(u_x^2)u_{xx}\big)_{xx} =& g(u_x^2)u_{xxxx} \\ &+ \big[4g''(u_x^2)u_x^2u_{xx}^2 + 2g'(u_x^2)u_{xx}^2 + 6g'(u_x^2)u_x u_{xxx}\big]u_{xx}.\end{aligned}$$

For any positive function g, the fourth order diffusion term never changes sign, and experiments show that with g as in (1) the second order term does not have a consistent sign. This suggests that forward-backward diffusion is not necessary for good edge detection, and that (9) is well-posed.

This paper addresses the well-posedness of two new denoising models proposed in [21]. While the analysis is focused on equations (10) and (11), it can be easily modified to apply to every other equation listed in the previous section, with the exception of (5). A theorem regarding the existence of local solutions to the new models is stated and proved in section 3. The proof uses techniques from the theory of maximal regularity, which does not require the validity of a maximum principle, and allows for enough flexibility to work with nonlocal operators such as fractional derivatives and to apply the results to a large class of equations. Key results from this theory which are used in the proof are given in the appendices. It should be observed that the well-posedness results come at a price. While in applications one would like to allow for L_∞ initial data, the techniques used here require smoother initial data. Such assumptions could be somewhat relaxed if weak formulations were to be used instead, but bounded measurable data could only be treated by approximating them by smooth ones and then going to the limit by means of any available a priori estimate.

2. New Models

We address the well-posedness of two modifications of the fourth order equation (5) proposed by You and Kaveh. One uses fractional derivatives to regularize the edge detector Δu in (5), and the other uses a fractional gradient as an edge detector, serving as a regularization of equation (9). Both equations (5) and (9) can benefit from the regularizing properties of fractional derivatives. (5) becomes well-posed when its edge detector $|\Delta u|$ is raised to a fractional power, and the fractional power on the edge detector $|\nabla u|$ in (9) alleviates a numerical artifact which is observed under certain parameters. The denoising effects of these two models are illustrated in figure 1.

Theorem 1 states the existence of local solutions to equations (10) and (11). These equations are considered with periodic boundary conditions on a

FIGURE 1. The denoising effects of equations (12) and (13), with g as in (1) and $\varepsilon = 0.1$. Top row: clean image (left); image corrupted with 20% Gaussian noise (right). Bottom row: image denoised with (12), $\varepsilon = 0.1$ (left); image denoised with (13), $\varepsilon = 0.1$ (right).

unit square, as this proved more effective in numerical experiments; Neumann conditions, on an arbitrary bounded domain with Lipschitz boundary, could also be considered, and the following theorem and proof could be reformulated to apply to that situation. The full equations being considered are

$$\begin{cases} u_t + \Delta\Big(g\big(|\nabla^{1-\varepsilon}u|^2\big)\Delta u\Big) = 0, & \text{in } \Omega \text{ for } t > 0,\\ u \text{ periodic}, & \text{for } t > 0,\\ u(0) = u_0, & \text{in } \Omega \text{ for } t = 0, \end{cases} \tag{12}$$

and

$$\begin{cases} u_t + \Delta\Big(g\big(\big[(-\Delta)^{1-\varepsilon}u\big]^2\big)\Delta u\Big) = 0, & \text{in } \Omega \text{ for } t > 0, \\ u \text{ periodic}, & \text{for } t > 0, \\ u(0) = u_0, & \text{in } \Omega \text{ for } t = 0. \end{cases} \tag{13}$$

This theorem is particularly powerful in that its proof does not depend on the choice of diffusivity function g, except for requiring that this function be smooth and positive. The theorem can also be easily modified to accommodate a nonzero fidelity term, i.e.,

$$u_t + \Delta\Big(g\big(|\nabla^{1-\varepsilon}u|^2\big)\Delta u\Big) = \lambda(u_0 - u),$$

provided it is sufficiently regular (see Theorem 3 and the accompanying discussion in appendix A). Such a fidelity term is often applied to insure that the smoothed image is not too dissimilar to the initial image. Numerical experiments on models (12) and (13) suggest, however, that such a term does not improve performance. Additionally, the techniques used in the proof of the theorem can be modified to be applied to equations of the form (6), (7), or (8). Consequently, Theorem 1 can be applied to a wide range of fourth order denoising models.

3. Main Theorem

Theorem 1. *Let $\Omega = [0,1]^n$, fix $\varepsilon \in (0,1)$ and $p > (4+n)/2\varepsilon$, and let $g \in BUC^\infty\big([0,\infty),(0,\infty)\big)$. For any $u_0 \in W^{4-4/p}_{p,\pi}(\Omega)$ there exists $T > 0$ such that equations (12) and (13) each possess a unique solution u on $[0,T)$ satisfying*

$$u \in W^1_p\big(0,T;L_p(\Omega)\big) \cap L_p\big(0,T;W^4_{p,\pi}(\Omega)\big).$$

Remarks. **(a)** *For equation (12), the theorem can be extended to allow $\varepsilon \in (-1,1)$ (thereby encompassing equation (9)) with the condition that $p > (4+n)/(1+\varepsilon)$.*
(b) *Initial conditions are taken in Slobodeckii spaces. These spaces are defined in the next section. For simplicity of notation, the dependence of these spaces on Ω will be dropped except for when not clear from the context.*

Theorem 1 is proved using techniques from the theory of maximal regularity. Background for the theory is postponed to appendix A, but some necessary preliminary definitions and results are given below.

3.1. Function Spaces and Fractional Derivatives

Let the usual Sobolev norm be defined by

$$\|u\|_{W^k_p}(\Omega) := \|u\|_{W^k_p} := \Big(\sum_{|\alpha|\le k} \|\partial^\alpha u\|^p_{L_p(\Omega)}\Big)^{1/p}, \quad k \in \mathbb{N},$$

where $\alpha \in \mathbb{N}^n$ is a multiindex, and define

$$[u]_{s,p} := \Big(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} d(x,y)\Big)^{1/p}, \quad 0 < s < 1.$$

Then for $s \in \mathbb{R}^+/\mathbb{N}$, the Slobodeckii spaces are the Banach spaces defined by

$$W_p^s := W_p^s(\Omega) := \{u \in W_p^{\lfloor s \rfloor} : [u]_{s,p} < \infty\},$$

equipped with the norm

$$\|u\|_{W_p^s} := \Big(\|u\|^p_{W_p^{\lfloor s \rfloor}} + \sum_{|\alpha| = \lfloor s \rfloor} [\partial^\alpha u]^p_{s - \lfloor s \rfloor, p}\Big)^{1/p},$$

where $\lfloor s \rfloor$ denotes the greatest integer less than or equal to s. Furthermore, for $s > 0$, define

$$W^s_{p,\pi} := W^s_{p,\pi}(\Omega) := \{u \in W_p^s : u \text{ is periodic on } \Omega\},$$

and observe that $L_p = L_{p,\pi}$.

The fractional gradient, under periodic boundary conditions, can be defined by first observing that

$$\partial_z = \mathcal{F}_z^{-1} \mathrm{diag}[(2\pi i k)_{k\in\mathbb{Z}}] \mathcal{F}_z,$$

where $\mathcal{F}_z$ is the partial Fourier transform with respect to $z = x, y$. Then the fractional partial derivative is defined by

$$\partial_z^\rho = \mathcal{F}_z^{-1} \mathrm{diag}\big[\big((2\pi k)^\rho e^{i\rho\frac{\pi}{2}\mathrm{sign}(k)}\big)_{k\in\mathbb{Z}}\big] \mathcal{F}_z,$$

for $\rho \in \mathbb{R}^+$. The fractional gradient is finally given by

$$\nabla^\rho = \begin{bmatrix} \partial_x^\rho \\ \partial_y^\rho \end{bmatrix}.$$

Exponents of the positive definite operator $-\Delta$ with periodic boundary conditions can be defined through its symbol:

$$(-\Delta)^\rho = \mathcal{F}^{-1} \mathrm{diag}\big[\big(4\pi^2 |k|^{2\rho}\big)_{k\in\mathbb{Z}^2}\big] \mathcal{F},$$

for $\rho \in \mathbb{R}^+$.

Since we are working with operators involving fractional derivatives, it is useful to have the Bessel potential spaces $H^s_{p,\pi}$ available. For $p \in (1,\infty), s > 0$, define

$$H^s_{p,\pi} := H^s_{p,\pi}(\Omega) := \{u \in L_p : \|u\|_{H^s_p} < \infty\},$$

where

$$\|u\|_{H^s_p} := \big\|\mathcal{F}^{-1} \mathrm{diag}\big[\big((1 + |k|^2)^{s/2}\big)_{k\in\mathbb{Z}^n}\big] \mathcal{F} u\big\|_{L_p}.$$

Fractional derivatives behave nicely on these spaces. It follows easily from the definitions that for $\rho > 0, s \in \mathbb{R}, p \in (1,\infty)$, and $j = 1, \dots, n$,

$$\partial^\rho_{x_j} \in \mathcal{L}\big(H^s_{p,\pi}, H^{s-\rho}_{p,\pi}\big),$$

and

$$(-\Delta)^\rho \in \mathcal{L}\big(H^s_{p,\pi}, H^{s-2\rho}_{p,\pi}\big).$$

Furthermore, Slobodeckii spaces can be embedded into Bessel spaces. It follows from results in [1, Chapter 5], along with [31, Theorems 1.3.3(e) and 1.10.3], that for any $s > \delta > 0$,

$$W^s_{p,\pi} \hookrightarrow H^{s-\delta}_{p,\pi},$$

and by a classical embedding theorem [31, Theorem 4.6.1(e)],

$$H^s_{p,\pi} \hookrightarrow C^{s-n/p}(\bar{\Omega}),$$

for $s > n/p$. The previous results imply that

$$\partial^\rho_{x_j} \in \mathcal{L}\big(W^s_{p,\pi}, C^{s-\rho-(\delta+n/p)}(\bar{\Omega})\big), \tag{14}$$

and

$$(-\Delta)^\rho \in \mathcal{L}\big(W^s_{p,\pi}, C^{s-2\rho-(\delta+n/p)}(\bar{\Omega})\big), \tag{15}$$

whenever the exponents on the C spaces are positive.

3.2. Main Lemma

The backbone of the proof of Theorem 1 is the following lemma, which is an amalgam of several results from the theory of maximal regularity. The proof of the lemma is postponed to appendix A.

Lemma 2. *Let $p \in [2, \infty)$ and Ω and u_0 as in Theorem* 1*, and consider the equation*

$$\begin{cases} u_t + A(u)u = 0, & \text{in } \Omega \text{ for } t > 0, \\ u \text{ periodic}, & \text{for } t > 0, \\ u(0) = u_0, & \text{in } \Omega \text{ for } t = 0, \end{cases} \tag{16}$$

where A is a fourth order elliptic nonlinear differential operator. Suppose that there exists a nonempty open subset $U \subset W^{4-4/p}_{p,\pi}$ containing u_0 such that the following hold:

(R1) $[v \mapsto A(v)] \in C^{1-}\big(U; \mathcal{L}(W^4_{p,\pi}, L_p)\big)$;

(R2) *for any $v \in U$ there exist $M > 0$ and $\theta \in [0, \pi/2)$ such that $A(v)$ is a uniformly (M, θ)-elliptic operator; and*

(R3) *for any $v \in U$ there exists $\rho \in (0, 1)$ such that the coefficients of $A(v) = \sum_{|\alpha| \le 4} a_\alpha \partial^\alpha$ are Hölder continuous with exponent ρ, that is, for all $|\alpha| \le 4, a_\alpha \in BUC^\rho(\Omega)$.*

Then there exist $T > 0$ and a unique function

$$u \in W^1_p(0, T; L_p) \cap L_p(0, T; W^4_{p,\pi})$$

satisfying (16) *for $t \in [0, T)$.*

Remarks. *The concept of a uniformly (M, θ)-elliptic operator is defined in* (29) *in appendix* B.

3.3. Existence Proof

Proof of Theorem 1. Let $\mathsf{A}u$ denote either $|\nabla^{1-\varepsilon} u|$ or $(-\Delta)^{1-\varepsilon} u$, according to whether we are considering (12) or (13). Let U be a bounded open subset of $W^{4-4/p}_{p,\pi}$ containing u_0. The operator of concern is

$$A(u)u = \Delta[\Phi_u \Delta u] = \Phi_u \Delta^2 u + \sum_{i=1}^n \big(\partial_{x_i} \Phi_u\big)\big(\partial_{x_i} \Delta u\big) + (\Delta \Phi_u)(\Delta u),$$

with

$$\Phi_u = g\big((\mathsf{A}u)^2\big).$$

Let $v \in W^{4-4/p}_{p,\pi}$ be any. Then

$$A(v)u = \Delta[\Phi_v \Delta u] = \sum_{|\alpha|\le 4} a_\alpha(x)\partial^\alpha u,$$

where $a_\alpha = a_{\alpha,v} : \Omega \mapsto \mathbb{C}$. (The a_α also depend on v, but for simplicity that dependence will be dropped from the notation when it is clear from the context.) When $n = 2$,

$$a_\alpha = \begin{cases} \Phi_v, & \alpha = (4,0),(0,4) \\ 2\Phi_v, & \alpha = (2,2) \\ 2\partial_{x_1}\Phi_v, & \alpha = (3,0),(1,2) \\ 2\partial_{x_2}\Phi_v, & \alpha = (2,1),(0,3) \\ \Delta\Phi_v, & \alpha = (2,0),(0,2) \\ 0, & \text{otherwise.} \end{cases}$$

(A similar calculation can be done when $n \neq 2$.) To prove the theorem, it suffices to show that A satisfies (R1), (R2), and (R3) from Lemma 2.

(R3): Given the calculation above, it is sufficient to show that there is $\rho \in (0,1)$ such that $\Phi_v \in C^{2+\rho}(\bar{\Omega})$. (14), (15), and the condition on p allow us to find $\rho \in (0,1)$ such that

$$(\mathsf{A}v)^2 \in C^{2+\rho}(\bar{\Omega}) \tag{17}$$

for either choice of A. It is discussed in [12] that the map

$$w \mapsto f(w), C^\beta(\bar{\Omega}) \to C^\beta(\bar{\Omega})$$

is, for any $\beta \in (0,1)$, well-defined, bounded, and uniformly Lipschitz continuous on bounded sets provided that the map $[x \mapsto f(x)]$ is twice continuously differentiable on its domain. Since $g \in BUC^\infty([0,\infty))$, it follows from this result and (17) that

$$g\big((\mathsf{A}v)^2\big), g'\big((\mathsf{A}v)^2\big), \text{ and } g''\big((\mathsf{A}v)^2\big) \in C^\rho(\bar{\Omega}). \tag{18}$$

Hölder spaces are algebras, so (R3) follows from (17) and (18).

(R2): The principal symbol of the operator $A(v)$ is given by

$$\begin{aligned} \mathcal{A}_\pi(x,\xi) &= \sum_{|\alpha|=4} a_\alpha(x)(i\xi)^\alpha \\ &= \Phi_v(\xi_1^4 + 2\xi_1^2\xi_2^2 + \xi_2^4) \\ &= \Phi_v|\xi|^4. \end{aligned}$$

Φ_v is positive by assumption, so in light of (17), there exist $c, C > 0$ such that

$$0 < c < \mathcal{A}_\pi(x,\xi) = \Phi_v(x) < C, \qquad (x,\xi) \in \Omega \times S^{n-1}.$$

Thus $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < c\} \subset \rho(\mathcal{A}_\pi)$, and so because $\mathcal{A}_\pi$ is real, there is $\theta \in (0, \pi/2)$ such that

$$\sigma(\mathcal{A}_\pi) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge c\} \cap \mathbb{R} \subset \Sigma_\theta := \{z \in \mathbb{C} : |\mathrm{arg} z| \le \theta\}.$$

It is also evident by (18) that there exists $M \geq C > 0$ such that

$$\max_{|\alpha|=4} \|a_\alpha\|_\infty \leq M.$$

Thus $A(v)$ satisfies the condition of (M,θ)-ellipticity (29), and so (R2) holds.

(R1): Pick $v, w \in U \subset W^{4-4/p}_{p,\pi}$. Then

$$\begin{aligned}
&\big\|A(v) - A(w)\big\|_{\mathcal{L}(W^4_{p,\pi}, L_p)} \\
&\quad = \sup_{\|u\|_{W^4_{p,\pi}}=1} \big\|\big(A(v)u - A(w)\big)u\big\|_{L_p} \\
&\quad \leq \sum_{|\alpha|\leq 4} \sup_{\|u\|_{W^4_{p,\pi}}=1} \big\|(a_{\alpha,v} - a_{\alpha,w})\partial^\alpha u\big\|_{L_p} \\
&\quad \leq \sum_{|\alpha|\leq 4} \sup_{\|u\|_{W^4_{p,\pi}}=1} \big\|a_{\alpha,v} - a_{\alpha,w}\big\|_\infty \big\|\partial^\alpha u\big\|_{L_p} \\
&\quad \leq \sum_{|\alpha|\leq 4} \big\|a_{\alpha,v} - a_{\alpha,w}\big\|_\infty
\end{aligned} \tag{19}$$

since $\|\partial^\alpha u\|_{L_p} \leq \|u\|_{W^4_{p,\pi}} = 1$. We show that

$$\|a_{\alpha,v} - a_{\alpha,w}\|_\infty \leq C\|v - w\|_{W^{4-4/p}_{p,\pi}}$$

only for $\alpha = (2,0)$, the most invloved case, as the details in other cases are similar. When $\alpha = (2,0)$ we have

$$\begin{aligned}
\|a_{\alpha,v} - a_{\alpha,w}\|_\infty &= \|\Delta g\big((\mathsf{A}v)^2\big) - \Delta g\big((\mathsf{A}w)^2\big)\|_\infty \\
&\leq \sum_{j=1}^n \|g''\big((\mathsf{A}v)^2\big)\|_\infty \|[\partial_{x_j}(\mathsf{A}v)^2]^2 - [\partial_{x_j}(\mathsf{A}w)^2]^2\|_\infty \\
&\qquad + \|[\partial_{x_j}(\mathsf{A}w)^2]^2\|_\infty \|g''\big((\mathsf{A}v)^2\big) - g''\big((\mathsf{A}w)^2\big)\|_\infty \\
&\qquad + \|g'\big((\mathsf{A}v)^2\big)\|_\infty \|\partial^2_{x_j}[(\mathsf{A}v)^2 - (\mathsf{A}w)^2]\|_\infty \\
&\qquad + \|\partial^2_{x_j}(\mathsf{A}w)^2\|_\infty \|g'\big((\mathsf{A}v)^2\big) - g'\big((\mathsf{A}w)^2\big)\|_\infty \\
&\leq \sum_{j=1}^n C\|\partial_{x_j}[(\mathsf{A}v)^2 + (\mathsf{A}w)^2]\|_\infty \|\partial_{x_j}[(\mathsf{A}v)^2 - (\mathsf{A}w)^2]\|_\infty \\
&\quad + C\|g'''\|_\infty \|(\mathsf{A}v)^2 - (\mathsf{A}w)^2\|_\infty \\
&\quad + C\|\partial^2_{x_j}[(\mathsf{A}v)^2 - (\mathsf{A}w)^2]\|_\infty \\
&\quad + C\|g''\|_\infty \|(\mathsf{A}v)^2 - (\mathsf{A}w)^2\|_\infty \\
&\leq C\sum_{j=1}^n \max_{u=v,w} \big\{\|\partial_{x_j}(\mathsf{A}u)^2\|_\infty\big\} \|\partial_{x_j}[(\mathsf{A}v)^2 - (\mathsf{A}w)^2]\|_\infty \\
&\quad + \|\partial^2_{x_j}[(\mathsf{A}v)^2 - (\mathsf{A}w)^2]\|_\infty \\
&\quad + \|(\mathsf{A}v)^2 - (\mathsf{A}w)^2\|_\infty.
\end{aligned}$$

The first inequality is obtained by computing the Laplacian and applying the triangle inequality. Since Ω is bounded, (17) and (18) imply the existence of a $C > 0$ such that

$$\|g''\big((\mathsf{A}v)^2\big)\|_\infty, \|[\partial_{x_i}(\mathsf{A}w)^2]^2\|_\infty, \\ \|g'\big((\mathsf{A}v)^2\big)\|_\infty, \text{ and } \|\partial^2_{x_i}(\mathsf{A}w)^2\|_\infty \le C.$$

The second and third inequalities follow from these bounds. All three terms in the last expression can be controlled by a constant times $\|v - w\|_{W^{4-4/p}_{p,\pi}}$ by taking advantage of the facts that by (14) and (15),

$$\partial^{1-\varepsilon}_{x_j}, (-\Delta)^{1-\varepsilon} \in \mathcal{L}\big(W^{4-4/p}_{p,\pi}, C^2(\bar\Omega)\big),$$

and that, since U is bounded, there exists a constant $M_U < \infty$ such that

$$M_U = \sup_{u \in U} \|u\|_{W^{4-4/p}_{p,\pi}}.$$

Indeed, for the first term, when $\mathsf{A}u = |\nabla^{1-\varepsilon}u|$, the maximum term can be bounded by

$$\begin{aligned} &\max_{u=v,w} \|\partial_{x_j}(\mathsf{A}u)^2\|_\infty = \max_{u=v,w} \|\partial_{x_j}|\nabla^{1-\varepsilon}u|^2\|_\infty \\ &\le \max_{u=v,w} \sum_{k=1}^n 2\|\partial^{1-\varepsilon}_{x_k}u\|_\infty \|\partial_{x_j}(\partial^{1-\varepsilon}_{x_k}u)\|_\infty \\ &\le \max_{u=v,w} \sum_{k=1}^n \|\partial^{1-\varepsilon}_{x_k}\|_{\mathcal{L}(W^{4-4/p}_{p,\pi}, C^2(\bar\Omega))} \|\partial_{x_j}\partial^{1-\varepsilon}_{x_k}\|_{\mathcal{L}(W^{4-4/p}_{p,\pi}, C^1(\bar\Omega))} \|u\|^2_{W^{4-4/p}_{p,\pi}} \\ &\le 2CM_U^2, \end{aligned}$$

and similarly when $\mathsf{A}u = (-\Delta)^{1-\varepsilon}$. Additionally, when A is the fractional Laplacian, we have

$$\begin{aligned} &\|\partial^2_{x_j}[(\mathsf{A}v)^2 - (\mathsf{A}w)^2]\|_\infty = \|\partial^2_{x_j}\big[\big((-\Delta)^{1-\varepsilon}v\big)^2 - (-\Delta)^{1-\varepsilon}w\big)^2\big]\|_\infty \\ &\quad \le \|\partial^2_{x_j}\|_{\mathcal{L}\left(C^2(\bar\Omega), C(\bar\Omega)\right)} \|(-\Delta)^{1-\varepsilon}(v+w)\|_{C^2(\bar\Omega)} \|(-\Delta)^{1-\varepsilon}(v-w)\|_{C^2(\bar\Omega)} \\ &\quad \le 2C\|(-\Delta)^{1-\varepsilon}\|^2_{\mathcal{L}\left(W^{4-4/p}_{p,\pi}, C^2(\bar\Omega)\right)} \max_{u=v,w}\big\{\|u\|_{W^{4-4/p}_{p,\pi}}\big\} \|v-w\|_{W^{4-4/p}_{p,\pi}} \\ &\quad \le CM_U\|v-w\|_{W^{4-4/p}_{p,\pi}}. \end{aligned}$$

The other necessary bounds can be found in the same manner, from which we conclude (R1). □

4. Conclusion

Several fourth order PDEs for image denoising have been analyzed. The existence of short time solutions to two newly proposed denoising models has been proved. The proof utilizes methods of maximal regularity which allow us to handle to nonlocal nonlinearity in the proposed equations, and which are flexible enough to be applied to other fourth order equations. Sufficient criteria for applying this method to other equations have been given.

Appendices

Appendix A. Maximal Regularity

Let $E_1 \hookrightarrow E_0$ be a densely embedded pair of Banach spaces, $p \in (1,\infty)$, and $T > 0$, and consider the system

$$\begin{cases} \dot{u}(t) + Bu(t) = f(t), & \text{on } (0,T), \\ u(0) = u_0, \end{cases} \tag{20}$$

where $B \in \mathcal{L}(E_1, E_0)$ and $f(t) \in L_p(0,T;E_0)$. We call

$$u \in W_p^1(0,T;E_0) \cap L_p(0,T;E_1)$$

a *strict solution* of (20) on $[0,T]$ if u satisfies (20) in the $L_p(0,T;E_0)$ sense. If B is the negative generator of an analytic semigroup and $f = 0$, (20) has a strict solution if and only if u_0 is in the trace space of $W_p^1(0,T;E_0) \cap L_p(0,T;E_1)$, that is, if

$$u_0 \in (E_0, E_1)_{1-\frac{1}{p},p} =: E_{1-\frac{1}{p},p},$$

where $(\cdot,\cdot)_{\theta,q}$ is the standard real interpolation functor (see [31]). That the trace space of $W_p^1(0,T;E_0) \cap L_p(0,T;E_1)$ is characterized by $E_{1-\frac{1}{p},p}$ follows from Theorem III.4.10.2 in [2], which also gives us the embedding

$$W_p^1(0,T;E_0) \cap L_p(0,T;E_1) \hookrightarrow C\big([0,T];E_{1-\frac{1}{p},p}\big).$$

See [2] as well for more information about trace spaces and maximal regularity in general.

$B \in \mathcal{L}(E_1, E_0)$ is said to have the property of maximal regularity, written $B \in \mathcal{MR}_p(E_0, E_1) =: \mathcal{MR}_p$, if, for every $f \in L_p(0,T;E_0)$ and $u_0 \in E_{1-\frac{1}{p},p}$, there exists a unique strict solution u of (20). In this case, the Open Mapping Theorem yields the existence of an $M > 0$, independent of f and u_0, such that

$$\begin{aligned} &\|u\|_{W_p^1(0,T;E_0)} + \|Bu(t)\|_{L_p(0,T;E_0)} \\ &\qquad \le M\big\{ \, \|f(t)\|_{L_p(0,T;E_0)} + \|u_0\|^p_{E_{1-\frac{1}{p},p}} \big\}. \end{aligned}$$

The following result about nonlinear equations of a form similar to (20), but with the operator B now depending, perhaps nonlocally, on u, is found in [13].

Theorem 3 (Clément and Li). *Let U be a nonempty open subset of $E_{1-\frac{1}{p},p}$. Suppose that*

$$A \in C^{1-}\big(U; \mathcal{L}(E_1, E_0)\big), \tag{21}$$

$$\phi \in C^{1-,1-}([0,T_0] \times U, E_0), \tag{22}$$

and

$$\psi \in L_p(0,T_0;E_0). \tag{23}$$

Let $u_0 \in U$. If $A(u_0) \in \mathcal{MR}_p$, then there exist $T \in (0, T_0]$ and a unique function $u \in W_p^1(0,T;E_0) \cap L_p(0,T;E_1)$ satisfying

$$\begin{cases} \dot{u}(t) + A\big(u(t)\big)u(t) = \phi\big(t, u(t)\big) + \psi(t) & \text{on } (0,T), \\ u(0) = u_0. \end{cases} \tag{24}$$

The crux of Theorem 3 is the requirement that $A(u_0) \in \mathcal{MR}_p$ for any $u_0 \in E_{1-\frac{1}{p},p}$. That is, existence of a solution to (24) can be found by instead studying a system of type (20) with $B = A(u_0)$. A well known result gives that in order for this to hold, it is necessary that $A(u_0)$ be the negative generator of an analytic semigroup. A sufficient condition is given by a theorem of Hieber and Prüß, which requires some preliminary definitions.

Suppose that for a domain Ω, an operator B generates an analytic C_0-semigroup $\mathcal{T}$ on $L_2(\Omega)$, where (Ω, μ, d) is a topological space such that there exist constants $c_1, c_2 > 0$ for which

$$|\mathbb{B}(x, 2\rho)| \leq c_1 |\mathbb{B}(x, \rho)|, \qquad \text{for all } x \in \Omega, \rho > 0, \tag{25}$$

and

$$\operatorname{ess\,sup}_{x \in \Omega} |\mathbb{B}(x, \rho)| \leq c_2 \operatorname{ess\,inf}_{x \in \Omega} |\mathbb{B}(x, \rho)|. \tag{26}$$

The doubling property (25) implies the existence of constants $c_3, \nu > 0$ such that

$$|\mathbb{B}(x, \lambda\rho)| \leq c_3 \lambda^{\nu} |\mathbb{B}(x, \rho)|, \qquad \text{for all } x \in \Omega, \lambda \geq 1. \tag{27}$$

Conditions (25) and (26) are satisfied on $\mathbb{R}^n$ as well as on bounded subsets of $\mathbb{R}^n$ with Lipschitz boundary.

Suppose also that $\mathcal{T}$ may be represented as an integral operator with kernel $K(\cdot,\cdot,\cdot) : (0,\infty) \times \Omega \times \Omega \to \mathbb{C}$ satisfying

$$(\mathcal{T}(t)f)(x) = \int_{\Omega} K(t,x,y) f(y) dy, \tag{28}$$

for a.e. $x, t > 0, f \in L_2(\Omega)$. K is said to satisfy a Poisson bound of order $k > 0$ if

$$|K(t,x,y)| \leq |\mathbb{B}(x, t^{1/k})|^{-1} h\left(\frac{d(x,y)^k}{t}\right),$$

for a.e. $x, y \in \Omega$ and all $t > 0$, where h is a bounded, decreasing, continuous and strictly positive function satisfying

$$\lim_{r \to \infty} r^{\nu+\delta} h(r^k) = 0$$

for some $\delta > 0$ and ν as in (27). The bounds on K imply the existence of a C_0-semigroup $\mathcal{T}_q$ on $L_q(\Omega)$ for any $q \in [1, \infty)$ satisfying

$$\mathcal{T}_q(t)f = \mathcal{T}(t)f, \qquad \text{for } f \in L_2(\Omega) \cap L_q(\Omega), t \geq 0.$$

Clearly $\mathcal{T}_q$ may also be represented by the kernel K as in (28). Denote by B_q the generator of $\mathcal{T}_q$.

The following theorem is from [24, Theorem 3.1].

Theorem 4 (Hieber and Prüß). *Let $1 < p, q < \infty$, and let (Ω, μ, d) be a topological space satisfying* (25) *and* (26)*. Let B be the negative generator of an analytic C_0-semigroup $\mathcal{T}$ on $L_2(\Omega)$ and assume that $\mathcal{T}$ may be represented by a kernel satisfying a Poisson bound of order $k > 0$. Then for each $f \in L_p\big(0,\infty; L_q(\Omega)\big)$ there exists a unique solution*

$$u \in W_p^1\big(0,\infty; L_q(\Omega)\big) \cap L_p\big(0,\infty; \mathrm{Dom}(B_q)\big)$$

of

$$\begin{cases} \dot{u} + Bu = f, & t \in (0,\infty) \\ \quad u(0) = 0, \end{cases}$$

in the $L_p(0,\infty; L_q(\Omega))$ sense.

Appendix B. Application to Fourth Order Diffusions

The two theorems discussed in appendix A can be applied to prove Lemma 2.

Proof of Lemma 2. Consider the abstract Cauchy problem (16). A is a fourth order differential operator, so set $E_0 = L_{p,\pi}(\Omega) = L_p$ and $E_1 = W^4_{p,\pi}$ By the discussion in appendix A, it is necessary to take the initial condition u_0 in the trace space $E_{1-\frac{1}{p},p}$, which, by results in [1, Chapter 5], can be characterized by

$$E_{1-\frac{1}{p},p} = (L_p, W^4_{p,\pi})_{1-\frac{1}{p},p} = W^{4-\frac{4}{p}}_{p,\pi}.$$

Given the choice of E_1, it is natural to consider $\mathcal{A}_p$, the L_p-realization of A, defined by

$$\begin{aligned} \mathrm{Dom}(\mathcal{A}_p) &:= W^4_{p,\pi} \\ \mathcal{A}_p f &:= Af \quad \text{for all } f \in W^4_{p,\pi}. \end{aligned}$$

In the notation of Theorem 3, $\phi \equiv \psi \equiv 0$, so (22) and (23) are satisfied. Theorem 4, with the aid of Theorem 3 and the result mentioned previously regarding strict solutions to (20) when $f = 0$, guarantees the existence of a time T such that (16) is satisfied on the interval $(0, T)$ (that is, existence of a short-time solution), provided there exists a nonempty open subset $U \subset E_{1-\frac{1}{p},p}$ for which the operator A satisfies the following requirements.

1. $A \in C^{1-}\big(U; \mathcal{L}(W^4_{p,\pi}, L_p)\big)$.
2. For any $v \in U, A(v)$ is the negative generator of an analytic C_0-semigroup $\mathcal{T}$ on $L_2(\Omega)$, and further, $\mathcal{T}$ may be represented by a kernel satisfying a Poisson bound of order $k > 0$.

Requirement 2 insures that the conditions of Theorem 4 are satisfied for the operator $A(v)$. Ω is bounded, so if $q \in [2,\infty)$, the generator A_q of the semigroup T_q discussed earlier is in fact $\mathcal{A}_q$, the L_q-realization of A. Then taking $q = p$, since requirement 2 insures that $-A(v)$ generates an analytic semigroup, Theorem 4 implies that for all $v \in U$,

$$A(v) \in \mathcal{MR}_p(L_p, W^4_{p,\pi}).$$

This, along with requirement 1, ensures that the conditions for Theorem 3 are satisfied, yielding the desired short-time existence of a solution to (16).

Requirement 1 is the same as (R1) from Lemma 2, and it is now shown that (R2) and (R3) together imply requirement 2. If (R3) holds, then the coefficients of the operator $A(v)$ are all Hölder continuous, that is, there is $\rho \in (0,1)$ such that $a_\alpha \in BUC^\rho(\Omega)$ for every $|\alpha| \leq 4$. Under these conditions, a result found in [4] gives that $A(v)$ is the negative generator of an analytic C_0-semigroup on $L_2(\Omega)$ if certain conditions on the principal symbol of $A(v)$ are satisfied. $A(v)$ is a fourth order linear differential operator, so it can be expressed as

$$A(u_0) = \sum_{|\alpha|\leq 4} a_\alpha \partial^\alpha,$$

where $a_\alpha : \Omega \to \mathbb{C}$. The principal symbol of $A(v)$ is

$$\mathcal{A}_\pi(x,\xi) := \sum_{|\alpha|=4} a_\alpha(x)(i\xi)^\alpha, \quad (x,\xi) \in \Omega \times \mathbb{R}^n.$$

For $M > 0$ and $\theta \in [0,\pi]$, $A(v)$ is said to be uniformly (M,θ)-elliptic if for all $x \in \Omega$ and $|\xi| = 1$ the following conditions hold:

$$\begin{cases} \max\limits_{|\alpha|=4} \|a_\alpha\|_\infty \leq M, \\ \sigma\big(\mathcal{A}_\pi(x,\xi)\big) \subset \big\{z \in \mathbb{C}/\{0\} \big| |\arg(z)| < \theta\big\}, \\ \big|[\mathcal{A}_\pi(x,\xi)]^{-1}\big| \leq M. \end{cases} \tag{29}$$

Here σ denotes the spectrum of an operator

$$\sigma\big(\mathcal{A}_\pi(x,\xi)\big) = \mathbb{C} \setminus \rho\big(\mathcal{A}_\pi(x,\xi)\big),$$

where

$$\rho\big(\mathcal{A}_\pi(x,\xi)\big) := \big\{\lambda \in \mathbb{C} : \big(\lambda I - \mathcal{A}_\pi(x,\xi)\big) \text{ is invertible}\big\}.$$

Condition (R2) requires that there is some $M > 0$ and $\theta \in [0,\pi/2)$ such that $A(v)$ is uniformly (M,θ)-elliptic. A theorem by Amann, Hieber, and Simonett [4, Corollary 9.5] implies that given (R2) and (R3), $A(v)$ is the negative generator of an analytic C_0-semigroup on $L_2(\Omega)$. Under these conditions, [16, Theorem 9.4.2] gives that $\mathcal{T}$, the semigroup generated by $A(u_0)$, can be represented by a kernel with a Poisson bound of order 4, with $h(r) = a\exp\{-br^{\frac{1}{3}}\}$ for some constants $a, b > 0$. Thus, requirement 2 is satisfied if (R2) and (R3) hold, and the argument is complete. $\square$

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