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Diffusion in Glassy Polymers: Sorption of a Finite Amount of Solvent

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Abstract. A family of 1-D moving boundary models describing the diffusion of a finite amount of a penetrant in a glassy polymer is studied. Local existence of a unique classical solution is obtained for a generic quasilinear model. Specific data are then chosen which can be found in the literature (cf. [6]) and global existence of the classical solution and its convergence to an equilibrium solution are proven. Finally a rigorous proof is provided for a formal perturbation argument proposed in [6] and used therein to estimate the rate of convergence of the solution towards the equilibrium.

1 The model

We consider a set of equations which describe the diffusion taking place when a polymer film is exposed to a finite amount of a smaller molecule capable of diffusing in the polymer. The model was introduced by Cohen and Erneux in [6] along with the corresponding model in the case when the polymer is exposed to a solvent kept at a constant given concentration. The latter model was already studied in [8]. As in that case one observes a sharp interface separating the two regions where the concentration of the penetrant is positive and where it is zero, respectively. This interface starts to move while a diffusive process takes place in the polymer region where the smaller molecule is already present. The following are the equations set up in [6] to describe this phenomenon:

$$C_T - DC_{XX} = 0\,, \qquad 0 < X < S(T)\,,\ T > 0\,, \tag{1.1}$$

$$C_X = 0\,, \qquad X = 0\,,\ T > 0\,, \tag{1.2}$$

$$-DC_X = (C+K)S_T\,, \qquad X = S(T)\,,\ T > 0\,, \tag{1.3}$$

$$S_T = k_1(C - C^*)^n\,, \qquad X = S(T)\,,\ T > 0\,, \tag{1.4}$$

$$S(0) = S_i > 0\,, \qquad T = 0\,. \tag{1.5}$$

$$C(\cdot, 0) = C_i\,, \qquad T = 0\,. \tag{1.6}$$

We used the same notation as in [6] where C and S are the unknown concentration of the solvent and the unknown front position, respectively. The constants appearing are all positive physical parameters. In particular C^* is the equilibrium concentration and D the diffusivity. Consider to have a slab of polymer with impermeable faces into which, initially, a finite fixed amount $C_i > C^*$ of penetrant is injected to a depth S_i. The impermeable membrane is either removed or becomes permeable when placed in a dissolving solution. Then (1.1) describes the diffusion in the region where the penetrant is present, whereas equation (1.2) models the impermeability of the left face of the slab. Equation (1.3) is a conservation law at the front and (1.4) describes the kinetic of the moving interface in terms of a phenomenological power law (where $n \geq 0$ is typically an integer). These equations were derived in connection with models for the formation and storage of simple swelling controlled drug release system without volume change (cf. [6]). The first models for diffusion in polymers go back to Alfrey-Gurnee-Lloyd [1] and Astarita-Sarti [5].

Remark 1.1. We observe that, as was pointed out in [6], the system admits a conservation relation which reads

$$\int_0^{S(T)} C\,dX + KS(T) \equiv Q = (C_i + K)S_i\,. \tag{1.7}$$

If (1.3) is substituted by the above relation in (1.1)–(1.6) an equivalent system is obtained. This fact will be used later. From (1.7) and (1.1)–(1.6) we deduce that the system admits the equilibrium solution $\left(C^*, \frac{Q}{C^*+K}\right)$.

After introducing dimensionless variables as in [6] we may rewrite the system as follows:

$$\dot{u} - \delta u_{xx} = 0\,, \qquad 0 < x < L(t)\,,\ t > 0 \tag{1.8}$$

$$u_x = 0\,, \qquad x = 0\,,\ t > 0\,, \tag{1.9}$$

$$-\delta u_x = (u + \epsilon^{-1})L'\,, \qquad x = L(t)\,,\ t > 0\,, \tag{1.10}$$

$$L' = u^n\,, \qquad x = L(t)\,,\ t > 0\,, \tag{1.11}$$

$$L(0) = \frac{\epsilon}{1+\epsilon}\,, \qquad t = 0 \tag{1.12}$$

$$u(\cdot, 0) = 1\,, \qquad 0 \leq x \leq \frac{\epsilon}{1+\epsilon}\,,\ t = 0 \tag{1.13}$$

where the parameters ϵ and δ are defined as

$$\epsilon = \frac{C_i - C^*}{C^* + K}\,,\ \delta = \frac{D}{k_1 Q}(C_i - C^*)^{1-n}\,.$$

Remark 1.2. In [6] the Neumann boundary condition (1.10) is actually replaced by the conservation relation, which now reads

$$\int_0^L u\,dx + \frac{L}{\epsilon} = 1\,, \tag{1.14}$$

to obtain an equivalent system. For our purposes it is more convenient to work with (1.10) instead.

In the new dimensionless variables the equilibrium solution becomes $(u, L) = (0, \epsilon)$. Lastly we transform the system to a fixed domain problem which is more easily dealt with. This is performed by the simple change of variables

$$(x,t) \mapsto (y,\tau) := (x/L(t), t)\,,\ \hat{u}(y,\tau) := u(L(\tau)y, \tau)\,.$$

One then obtains the system

$$\begin{aligned}
\dot{\hat{u}} - \frac{\delta}{L^2(t)}\hat{u}_{yy} &= y\frac{L'(t)}{L(t)}\hat{u}_y\,, && 0 < y < 1\,,\ t > 0 && (1.15)\\
\hat{u}_y &= 0\,, && y = 0\,,\ t > 0\,, && (1.16)\\
-\delta\hat{u}_y &= (\hat{u} + \epsilon^{-1})L'(t)L(t)\,, && y = 1\,,\ t > 0\,, && (1.17)\\
L'(t) &= \hat{u}^n(1,t)\,, && y = 1\,,\ t > 0\,, && (1.18)\\
L(0) &= \frac{\epsilon}{1+\epsilon}\,, && t = 0 && (1.19)\\
\hat{u}(\cdot, 0) &= 1\,, && 0 \le y \le 1\,,\ t = 0\,. && (1.20)
\end{aligned}$$

We can now formulate the main results concerning the model.

Theorem 1.3. *The system* (1.15)–(1.20) *possesses a unique classical global solution* (u, L). *Moreover it converges towards the equilibrium solution* $(0, \epsilon)$ *in the sense that*

$$\big(\hat{u}(t), L(t)\big) \longrightarrow (0, \epsilon) \ \text{as } t \to \infty$$

in the topology of $C^\sigma(0,1) \times \mathbb{R}$ *for some* $\sigma \in (0,1)$.

By $C^\sigma(0,1)$ we mean the Banach space of Hölder continuous functions on $[0,1]$ with norm

$$\|u\| := \|u\|_\infty + \sup_{x \ne y} \frac{|u(x) - u(y)|}{|x-y|^\sigma}$$

for $u \in C^\sigma(0,1)$. To determine the rate of convergence of the front towards its equilibrium position the authors in [6] perform some formal asymptotics. In particular they provide evidence to believe that the solutions $(\hat{u}_\delta, L_\delta)$ converge for $\delta \to \infty$ towards the solution (u_∞, L_∞) (only depending on time) of

$$L'_\infty = u^n_\infty\,,\ \ L_\infty(0) = \frac{\epsilon}{1+\epsilon}\,, \tag{1.21}$$

$$\big(u_\infty + \frac{1}{\epsilon}\big)L_\infty = 1 \tag{1.22}$$

A rigorous proof of this suggestion can be given. In the following theorem we determine the system of ODEs which is satisfied by the limiting function (u_∞, L_∞) and which can be showed to be equivalent to the above system suggested by Cohen and Erneux.

Theorem 1.4. *Let $(\hat{u}_\delta, L_\delta)$ be the solution of (1.15)–(1.20). Then we have*

$$(\hat{u}_\delta(t), L_\delta(t)) \longrightarrow (u_\infty(t), L_\infty(t))\,, \ \text{as } \delta \to 0$$

in the topology of $C^\sigma\big(I, W^1_p(0,1)\big) \times C^\sigma(I)$ for each compact subinterval I of $(0,\infty)$, where (u_∞, L_∞) is the solution of the following system of ODEs

$$\dot{u}_\infty = -\frac{u^n_\infty}{L_\infty}(\frac{1}{\epsilon} + u_\infty) \tag{1.23}$$

$$L'_\infty = u^n_\infty \tag{1.24}$$

$$u_\infty(0) = 1 \tag{1.25}$$

$$L_\infty(0) = \frac{\epsilon}{1+\epsilon} \tag{1.26}$$

The convergence is uniform on compact subintervals of $(0,\infty)$.

Corollary 1.5. *The systems (1.21)–(1.22) and (1.23)–(1.26) are equivalent. The solution (u_∞, L_∞) converges towards the equilibrium solution $(0,\epsilon)$ as $t \to \infty$.*

Remark 1.6. Since $(\hat{u}_\delta, L_\delta)$ converges towards (u_∞, L_∞) as $\delta \to \infty$, the behaviour of L_δ at infinity can be compared to that of L_∞ for δ large enough. This is precisely what Cohen and Erneux do in their paper. The proofs of the above results are postponed to the following sections of the paper. First we need to introduce some useful notation and to fix the functional setting in which to work.

2 Local existence in the quasilinear case

In this section we shall prove local solvabilty of a generic quasilinear system of the type (1.15)–(1.20). Local solvabilty of the latter will follow as a corollary. Consider

$$\dot{\hat{u}} - \partial_y\big(a(\hat{u},L)\partial_y\hat{u}\big) = f(\hat{u},\partial_y\hat{u},L,L')\,, \quad 0<y<1\,,\ \ t>0 \tag{2.1}$$

$$a(\hat{u},L)\partial_y\hat{u} = g(\hat{u},L,L')\,, \qquad y=0\,,\ \ t>0\,, \tag{2.2}$$

$$-a(\hat{u},L)\partial_y\hat{u} = h(\hat{u},L,L')\,, \qquad y=1\,,\ \ t>0\,, \tag{2.3}$$

$$L'(t) = k(\hat{u},L)\,, \qquad y=1\,,\ \ t>0\,, \tag{2.4}$$

$$L(0) = L_0 > 0\,, \qquad t=0 \tag{2.5}$$

$$\hat{u}(\cdot,0) = \hat{u}_0\,, \qquad 0\le y\le 1\,,\ \ t=0 \tag{2.6}$$

Multiplying by a test function $\varphi \in W^1(0,1)$ and integrating by parts the second term on the left-hand side of (2.1) and making use of the boundary conditions

(2.2) and (2.3), we obtain

$$\langle \hat{u}_t, \varphi\rangle + \langle a\partial_y \hat{u}, \partial_y \varphi\rangle = \langle f, \varphi\rangle - \langle g, \varphi(0)\rangle - \langle h, \varphi(1)\rangle\,,\ \varphi \in W^1(0,1)\,, \tag{2.7}$$

$$\hat{u}(0) = \hat{u}_0\,, \tag{2.8}$$

$$L' = k(\hat{u}, L)\,,\ \ L(0) = L_0 \tag{2.9}$$

where in the first equation we omitted the arguments of the nonlinearities involved. We shall now interpret the above system as an abstract evolution equation in an appropriate Banach space. To this end let $W_p^{2s}(0,1)$ be the space of the Sobolev-Slobodevskii scale for positive s and

$$W_p^{2s}(0,1) := \big(W_{p'}^{-2s}(0,1)\big)'\,,$$

for negative s. For our purposes we shall only consider these spaces for $2s \in (-2+1/p, 1+1/p)$. We observe that

$$W_p^{2s}(0,1) \overset{d}{\hookrightarrow} L_p \overset{d}{\hookrightarrow} W_p^{-2r}(0,1) \tag{2.10}$$

for $2s \in (0, 1+1/p)$ and $2r \in (-2+1/p, 0)$. By

$$A(\hat{u}, L) \in \mathcal{L}\big(W_p^{2s}(0,1), W_p^{2s-2}(0,1)\big)$$

we denote the operator induced by the Dirichlet form

$$\mathbf{a}(\hat{u}, L) \in \mathcal{L}\big(W_p^{2s}(0,1) \times W_{p'}^{2-2s}(0,1), \mathbb{R}\big)$$

defined through

$$\mathbf{a}(\hat{u}, L)(v,w) := \langle a(\hat{u}, L)\partial_y v, \partial_y w\rangle\,,\ \ v \in W_p^{2s}(0,1)\,,\ w \in W_{p'}^{2-2s}(0,1)\,.$$

Moreover we define F by

$$F(\hat{u}, L) = \langle f(\hat{u}, \partial_y \hat{u}, L, K), \cdot\rangle + \langle \gamma_0' g(\gamma_0 \hat{u}, L, K), \cdot\rangle + \langle \gamma_1' h(\gamma_1 \hat{u}, L, K(\hat{u}, L)), \cdot\rangle$$

and K by

$$K(\hat{u}, L) = k(\gamma_1 \hat{u}, L)\,.$$

By γ_j we denoted the trace operator at the boundary point j. Using these definitions we may rewrite the system as follows:

$$\dot{\hat{u}} + A(\hat{u}, L)\hat{u} = F(\hat{u}, L)\,,\ \ t > 0\,,\ \text{in } W_p^{2s-2}(0,1)\,, \tag{2.11}$$

$$L' = K(\hat{u}, L)\,,\ \ t > 0\,, \tag{2.12}$$

$$\hat{u}(0) = \hat{u}_0\,, \tag{2.13}$$

$$L(0) = L_0\,. \tag{2.14}$$

The latter may, in its turn, be viewed as

$$\dot{V} + \mathbb{A}V = \mathbb{F}(V)\,,\;\; t > 0\,, \tag{2.15}$$

$$V(0) = V_0\,, \tag{2.16}$$

in the Banach space $\mathbb{E}_0$ setting

$$V = (\hat{u}, L)\,,\;\; \mathbb{E}_0 = W_p^{2s-2}(0,1) \times \mathbb{R}\,, \tag{2.17}$$

$$\mathbb{A}(V) = \begin{bmatrix} A(\hat{u}, L) & 0 \\ 0 & 0 \end{bmatrix}\,,\;\; \mathbb{F}(V) = \begin{bmatrix} F(\hat{u}, L) \\ K(\hat{u}, L) \end{bmatrix}\,. \tag{2.18}$$

We conculde that, by the general Theorem 12.1 in [3], there exists a unique local solution V on an interval $J = [0, T]$ of the latter evolution equation with

$$(\hat{u}, L) \in C^1\big(\dot{J}, \mathbb{E}_0\big) \cap C\big(\dot{J}, \mathbb{E}_1\big) \cap C^\rho\big(\dot{J}, \mathbb{X}_\alpha\big)\,,$$

for some $\rho \in (0,1)$ and with $\dot{J} := (0, T]$, provided

(A1) $\mathbb{A} \in C^{1-}\big(\mathbb{X}_\alpha, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0)\big)$.

(A2) $\mathbb{F} \in C^{1-}\big(\mathbb{X}_\alpha, \mathbb{E}_\beta\big)$

for some $0 < \beta < \alpha < 1$ and $\mathbb{X}_\alpha \overset{o}{\subset} \mathbb{E}_\alpha$. Some explanations about the notation are in order. By $\mathbb{E}_\alpha$ we denote the interpolation space obtained by the standard real interpolation functor $(\cdot,\cdot)_{\alpha,p}$. With $\overset{o}{\subset}$ we mean "included as an open subset". Lastly

$$\mathcal{H}\big(\mathbb{E}_1, \mathbb{E}_0\big)$$

denotes the set of all negative generators of analytic C_0-semigroups $-\mathbb{A}$ which satisfy

$$\mathrm{dom}(\mathbb{A}) \doteq \mathbb{E}_1\,,$$

where $\mathbb{E}_1 := W_p^{2s}(0,1) \times \mathbb{R}$ and $\doteq$ means "equal except (possibly) for an equivalent norm". The set $\mathcal{H}\big(\mathbb{E}_1, \mathbb{E}_0\big)$ is given the topology induced by $\mathcal{L}\big(\mathbb{E}_1, \mathbb{E}_0\big)$, the space of linear and continuous operators from $\mathbb{E}_1$ to $\mathbb{E}_0$.

Remark 2.1. It is to be pointed out that one has

$$\mathbb{E}_\alpha \doteq W_p^{2s-2+2\alpha}(0,1) \times \mathbb{R}$$

for $\alpha \in (0,1)$ except for $\alpha = 1 - s$.

The following lemma gives sufficient conditions for the validity of (A1) and (A2).

Lemma 2.2. *Assume the following hypotheses are satisfied:*

(i) $a \in C^{2-}\big(\mathcal{O}, [\underline{a}, \overline{a}]\big)$ *for some* $\mathcal{O} \overset{o}{\subset} \mathbb{R}^2$ *and some* $0 < \underline{a} < \overline{a}$.

(ii) $f \in C^{1-}\big(\mathbb{R}^4, \mathbb{R}\big)$ *such that* $\partial_2 f$ *is polynomially bounded in the second variable.*

(iii) $h, g \in C^{1-}\big(\mathbb{R}^3, \mathbb{R}\big)$.

(iv) $k \in C^{1-}\big(\mathbb{R}^2, \mathbb{R}\big)$.

Then, setting

$$\mathbb{E}_0 = W_p^{2s-2}(0,1) \times \mathbb{R}\,,\ \mathbb{E}_1 = W_p^{2s}(0,1) \times \mathbb{R}$$
$$\textit{for an appropriate choice of } 2s \in (1/p, 1+1/p)$$
$$\mathbb{E}_\beta = E_\beta \times \mathbb{R}\,,\ \mathbb{X}_\alpha = X_\alpha \times pr_2\mathcal{O} \ \textit{and}$$
$$X_\alpha = E_\alpha \cap W_p^{2s-2+2\alpha}\big((0,1), pr_1\mathcal{O}\big) \ \textit{for } 2s-2+2\alpha > 1/p\,,$$

the conditions (A1) and (A2) are satisfied (provided of course α and β are not singular values in the sense of Remark 2.1).

Proof. (α) We start by considering $\mathbb{F}$. We recall

$$\begin{aligned} F(\hat u, L) = \langle f(\hat u, \partial_y \hat u, L, K), \cdot\rangle + \langle \gamma_0' g(\gamma_0 \hat u, L, K), \cdot\rangle + \langle \gamma_1' h(\gamma_1 \hat u, L, K), \cdot\rangle \\ = F_1 + F_2 + F_3 \end{aligned}$$

for K defined as $K(\hat u, L) = k(\gamma_1 \hat u, L)$. First observe that the mapping

$$\big[(\hat u, L) \mapsto (\hat u, \hat u_y, L)\big]$$

maps $\mathbb{X}_\alpha$ into

$$X_\alpha \times W_p^{2s-3+2\alpha}(0,1) \times pr_1\mathcal{O} \hookrightarrow C\big([0,1]\big) \times L_q(0,1) \times pr_1\mathcal{O}\,,$$

for $q = 1/\varepsilon$, if $2s$ is chosen such that $2s - 3 + 2\alpha = 1/p - \varepsilon$, which is possible since α can be chosen arbitrary close to 1 and $2s$ to $1 + 1/p$. Lemma 14.2 in [3] concerning Nemitskii operators then implies that

$$F_1 \in C_b^{1-}\big(\mathbb{X}_\alpha, L_p(0,1)\big) \hookrightarrow C^{1-}\big(\mathbb{X}_\alpha, E_\beta\big)$$

since $L_p \hookrightarrow W_p^{2s-2+2\beta} = E_\beta$ provided β is chosen small enough as follows from Remark 2.1. As to F_2 and F_3 we first observe that

$$\gamma_j \in \mathcal{L}\big(W_{p'}^r, \mathbb{R}\big)\,,\ 1 + 1/p' > r > 1/p'$$

which implies

$$\gamma_j' \in \mathcal{L}\big(\mathbb{R}, W_p^{-r}\big)\,.$$

Thus, fixing r and choosing β small enough we infer from

$$\mathbb{X}_\alpha \overset{(\gamma_j, id)}{\longrightarrow} \mathcal{O} \overset{g,h,k \in C^{1-}}{\longrightarrow} \mathbb{R} \overset{\gamma_j'}{\longrightarrow} E_\beta$$

that

$$F_2\,,\ F_3 \in C^{1-}(\mathbb{X}_\alpha, E_\beta)\,.$$

In the above diagram one has of course to set $j = 0$ if choosing g for the second map or $j = 1$ if choosing h or k. Hereby we also showed that

$$K \in C^{1-}(\mathbb{X}_\alpha, \mathbb{R})\,.$$

(β) We infer again from Lemma 14.2 in [3] that

$$\big[(\hat{u},L)\mapsto a(\hat{u},L)\big]\in C^{1-}\big(C^\sigma([0,1],\mathcal{O}),C^\sigma([0,1],[\underline{a},\overline{a}])\big)$$

provided $\sigma\in[0,1)\cup\{1-\}$. On the other hand we see

$$\begin{array}{rcl}\mathbb{X}_\alpha \overset{C^{1-}}{\hookrightarrow} & C^\sigma([0,1])\times pr_2\mathcal{O} & \overset{C^{1-}}{\longrightarrow}\mathcal{H}\big(\mathbb{E}_1,\mathbb{E}_0\big)\\ V\mapsto & a(\hat{u},L) & \mapsto -\mathbb{A}(\hat{u},L)\end{array}$$

for an appropriate choice of $\sigma\in(0,1)$ (cf. Paragraph 8 in [3]). The embedding follows from the fact that

$$X_\alpha \overset{o}{\subset} E_\alpha \hookrightarrow C^{2s-2+2\alpha-1/p}\big([0,1]\big)$$

and that $2s$ may be chosen arbitrary close to $1+1/p$ and α to 1, which, in its turn, implies that $2s-2+2\alpha-1/p$ can be made almost 1. Thus

$$\mathbb{A}\in C^{1-}\big(\mathbb{X}_\alpha,\mathcal{H}(\mathbb{E}_1,\mathbb{E}_0)\big)\,.$$

The fact that $\mathbb{A}(\hat{u},L)\in\mathcal{H}\big(\mathbb{E}_1,\mathbb{E}_0\big)$ is implied by

$$\underline{a}\le a(\hat{u},L)\le\overline{a}\,,$$

the regularity of $[y\mapsto a(\hat{u},L)(y)]$ and the generation Theorem 8.5 in [3].
The proof is now complete. $\square$

Corollary 2.3. *Assume that the hypotheses of the above lemma are satisfied. Then the system* (2.1)–(2.6) *possesses for each $V_0=(\hat{u}_0,L_0)\in\mathbb{X}_\alpha$ a unique local weak solution $V=(\hat{u},L)$ (on an interval J) which enjoys the following regularity:*

$$\hat{u}\in C(\dot{J},E_1)\cap C^1(\dot{J},E_0)\cap C(J,X_\alpha)\,,\quad L\in C^1(J,\mathbb{R})$$

Proof. The assertion are a direct consequence of the previous lemma and Theorem 12.1 in [3]. $\square$

Remark 2.4. If all data are smooth it can be shown that the local weak solution is in fact a classical solution. The proof is based on boot-strapping arguments as it is described in Paragraph 14 of [3].

Remark 2.5. It follows also from [3, Thm. 12.1] that the sistem (2.1)–(2.4) generates a local semiflow on $\mathbb{X}_\alpha$.

Corollary 2.6. *The special system* (1.15)–(1.20) *possesses a unique local classical solution $(\hat{u},L)$.*

Proof. In this case we set

$$\mathcal{O}:=(0,\infty)\times(L_0/2,2L_0)$$

for $L_0:=\frac{\epsilon}{1+\epsilon}$ and easily check that the following are satisfied

(i) $a \in C^{2-}\big(\mathcal{O}, [\frac{\delta}{2L_0^2}, \frac{2\delta}{L_0^2}]\big)$.

(ii) $f \in C^{1-}\big(\mathbb{R}^4, \mathbb{R}\big)$ and is polynomially bounded in second variable.

(iii) $g, h \in C^{1-}\big(\mathbb{R}^3, \mathbb{R}\big)$.

(iv) $k \in C^{1-}\big(\mathbb{R}^2, \mathbb{R}\big)$.

Thus we are in the situation of Corollary 2.3 and conclude by Remark 2.4 that the assertion is true. □

Remark 2.7. It follows from the theory summarized in [3] that the solution of the system under consideration may be seeked as the unique fixed point of the variation-of-constants-formula, that is, of

$$V(t) = \mathbb{U}_V(t,0)V_0 + \int_0^t \mathbb{U}_V(t,\tau)\mathbb{F}\big(V(\tau)\big)\,d\tau$$

for the evolution operator $\mathbb{U}_V$ generated by the family $-\mathbb{A}(V)$ (for fixed V chosen in an appropriate set). In particular for the first component of V we have

$$\hat{u}(t) = U_{u,L}(t,0)\hat{u}_0 + \int_0^t U_{\hat{u},L}(t,\tau)F\big(\hat{u}(\tau), L(\tau)\big)\,d\tau$$

for the propagator generated by $-A(\hat{u}, L)$.

Remark 2.8. In the special case of equations (1.15)–(1.20) we can modify the Dirichlet form to

$$\mathbf{a}(\hat{u}, L)(v, w) := \frac{\delta}{L^2}\langle \partial_y v, \partial_y w\rangle - \frac{\hat{u}^n(1,t)}{L(t)}\langle y\partial_y v, w\rangle\,, \qquad v \in W_p^{2s}(0,1)\,,\ w \in W_{p'}^{2-2s}(0,1) \tag{2.19}$$

without loosing the generation properties of the previously defined Dirichlet form (the new term is a lower order perturbation) and the nonlinearity to

$$F(\hat{u}, L) = \frac{\gamma_1\hat{u}^n}{L}\langle \gamma_1'(\gamma_1\hat{u}^n + \epsilon^{-1}), \cdot\rangle$$

improving its mapping properties (the first derivative of u is not present in the new nonlinearity). This allows to apply again [3, Thm. 12.1] to obtain a local semiflow on $X := W_p^r \cap \big\{u \in C[0,1] \mid u \geq 0\big\} \times \mathbb{R}^+$ for $r \in (1/p, 1)$, which will be important in the next section.

3 Global existence and asymptotic behaviour

In this section we shall prove that the solution $(\hat{u}, L)$ of (1.15)–(1.20) exists globally and converges to the equilibrium solution $(0, \epsilon)$. To this end we need first a couple of lemmas.

Lemma 3.1. *Let* (u, L) *be the solution of* (1.8)–(1.13)*. Then*

$$\begin{aligned} 0 <& u(x,t) \leq 1\,,\ x \in [0, L(t)]\,,\\ 0 <& L'(t) \leq 1\,,\ t \geq 0\,,\\ \frac{\epsilon}{\epsilon+1} \leq& L(t) \leq \epsilon\,,\ t \geq 0\,. \end{aligned}$$

Remark 3.2. It is straightforward that any estimate obtained for u immediately translates into one for $\hat{u}$.

Proof. Since u satisfies (1.8)–(1.13) it follows from (1.8) that u takes its minimum/maximum on the parabolic boundary

$$t = 0 \text{ and } x \in [0, L_0] \text{ or } t > 0 \text{ and } x = 0, L(t)\,.$$

Suppose now that u becomes 0 at some point, then of course, that must occur at $x = 0$ or at $x = L(t)$ for a strictly positive time. As consequence of (1.9) this cannot happen at $x = 0$. From this we can conclude by (1.11) that $L'(t) = 0$, which entails together with the conservation relation (1.14) that $u(t) \equiv 0$. From this and the maximum principle we infer that $u(\cdot, t) \equiv 0$ for all $t \leq t_0$, which contradicts (1.13). Thus we conclude that $u > 0$. Next assume that u takes a maximum which is strictly larger than 1 at some time $t_0 > 0$. This must happen on the parabolic boundary. Once more it can be excluded by (1.9) that this occurs at $x = 0$, so, by (1.11), we see that

$$L'(t_0) = u^n(L(t_0), t_0) > 1\,.$$

But the strong maximum principle requires that

$$0 > -\delta u_x(L(t_0), t_0) = \big(u(L(t_0), t_0) + 1/\epsilon\big)L'(t_0)\,,$$

thus we have to conclude that

$$u(L(t_0), t_0) + 1/\epsilon < 0\,,$$

which is of course impossible. So we have to reject the assumption and we obtain the desired inequality $u \leq 1$. From the inequalities for u and (1.11) it follows that $0 < L'(t) \leq 1$. Lastly, using the bounds obtained for u and the conservation relation (1.14), it is easily seen that the last assertion concerning L is valid. This finishes the proof. □

Next we establish bounds for u_x.

Lemma 3.3. *The solution u of* (1.8)–(1.13) *satisfies*

$$-\frac{1+\epsilon}{\delta\epsilon} \le u_x(x,t) \le 0\,,\; 0 < x < L(t)\,,\;\; t > 0\,.$$

Proof. It is easily seen that $v := u_x$ satisfies

$$\begin{aligned} \dot{v} - \delta v_{xx} &= 0\,, &\qquad& 0 < x < L(t)\,,\;\; t > 0\,,\\ v &= 0\,, && x = 0\,,\;\; t > 0\,, \end{aligned}$$

From the validity of the first equation we infer that v attains its maximum and its minimum on the parabolic boundary. There we have $v \equiv 0$ for $t = 0$, $v = 0$ for $x = 0$ and, by the previous lemma and the boundary condition (1.10),

$$-\frac{\epsilon+1}{\epsilon\delta} \le v(L(t),t) \le 0\,.$$

Since we have obtained bounds on the whole parabolic boundary the claim follows. □

As a consequence of the previous lemmas we obtain the following

Corollary 3.4. *If* $(\hat{u}, L)$ *is a weak solution of the system* (1.15)–(1.20) *then it exists globally.*

Proof. Corollary 2.6 imply that the solution is classical. Further we know from Remark 2.8 that the system under consideration (in the transformed variables, that is, in the fixed domain) generates a local semiflow φ on

$$X := W_p^r \cap \big\{u \in C[0,1] \,|\, u \ge 0\big\} \times \mathbb{R}^+$$

for $r \in (1/p, 1)$. The bounds obtained in the previous lemmas imply that $\big\{(u(t), L(t)) \;:\; t \ge 0\big\}$ is relatively compact in X and that $(u(t), L(t))$ can not converge in finite time to the boundary of X, thus the solution must exist globally. □

Remark 3.5. The system (1.15)–(1.20) is equivalent to the system obtained by replacing the boundary condition (1.17) by

$$\Big(\int_0^1 \hat{u}\, dy + \epsilon^{-1}\Big) L = 1\,.$$

Proof. This follows from the discussion following Remark 1.1. □

Proposition 3.6. *The solution* $(\hat{u}, L)$ *of the initial boundary value problem* (1.15)–(1.20) *converges towards the equilibrium solution* $(0, \epsilon)$ *in the topology of* $C^\sigma[0,1] \times \mathbb{R}$ *for an appropriate* $\sigma \in (0,1)$.

Proof. We divide the proof into two steps corresponding to the two assertions of the proposition.
(α) If $(\hat{u}, L)$ is a solution of the stationary problem then $\hat{u} = 0$ and the conservation relation (1.14) implies $\frac{L}{\epsilon} = 1$. The convergence remains to be proved.
(β) As noticed in Remark 2.5 the system generates a local semiflow φ on the set X specified there. It is further a consequence of the conservation relation that

$$V(\hat{u}, L) := \int_0^1 \hat{u}\, dy$$

is a Lyapunov function for φ. In fact,

$$V(\hat{u}, L)'(t) = -\frac{L'(t)}{L^2(t)} \leq 0\,.$$

In particular it is a Lyapunov function on $\overline{M}$ where $M := \gamma^+(\hat{u}_0, L_0)$ is the orbit starting at $(\hat{u}_0, L_0)$. The previous lemmas imply that M is relatively compact in X and a simple maximum principle argument that

$$\{(0, \epsilon)\} = \{x \in \overline{M} \,:\, \lim_{t \to 0} \frac{1}{t}(V(t \cdot x) - V(x)) = 0\}$$

where $t \cdot x$ is the value at time t of the solution to the initial datum x. The assertion is thus a consequence of La Salle principle (cf. for instance [2, Cor. 18.4]) and the embedding $X \hookrightarrow C^\sigma[0,1] \times \mathbb{R}$, which is valid for $\sigma > 0$ small enough. $\square$

4 Rigorous proof of the formal asymptotics

This last section is devoted to the formal asymptotics performed in [6]. The next theorem provides a rigorous proof for it which uses a result about singular perturbations of non autonomous Cauchy problems. We shall postpone the proof of the latter to the appendix

Theorem 4.1. *Let $(\hat{u}^\delta, L_\delta)$ be the unique solution of (1.15)–(1.20) for each $\delta > 0$. Then $(\hat{u}^\delta, L_\delta)$ converges towards (u_∞, L_∞) in the topology of*

$$C^\rho\big(I, W_p^1(0,1)\big) \times C^\rho(I)$$

for an appropriate $\rho \in (0,1)$ and for each compact subinterval I of $(0, \infty)$. The function (u_∞, L_∞) is the unique solution of the following system of ODEs:

$$\dot{u}_\infty = -\frac{u_\infty^n}{L_\infty}(\frac{1}{\epsilon} + u_\infty)\,, \qquad u_\infty(0) = 1\,, \tag{4.1}$$

$$L'_\infty = u_\infty^n\,, \qquad L_\infty(0) = \frac{\epsilon}{1+\epsilon}\,. \tag{4.2}$$

Proof. We recall that in the functional setting introduced in Section 2 (see in particular Remark 2.7) it is possible to represent the solution $(\hat{u}^\delta, L_\delta)$ of the system under consideration by the variation-of-constants-formula, that is, by

$$\begin{aligned}\hat{u}_\delta(t) &= T_N\Big(\int_0^t \frac{\delta}{L^2(\sigma)}\,d\sigma\Big)u_0 + \int_0^t T_N\Big(\int_\tau^t \frac{\delta}{L^2(\sigma)}\,d\sigma\Big)F_\delta(\tau)\,d\tau\\ L_\delta(t) &= L_0 + \int_0^t \hat{u}^n(1,\tau)\,d\tau\,.\end{aligned}$$

By T_N we denoted the semigroup generated by $-A_N = (-\partial_{xx}, \gamma_0\partial_x, \gamma_1\partial_x)$, the Neumann Laplacian, on the phase space $E_0 = W_p^{2s-2}(0,1)$ for an appropriate $s \in (1/p, 1+1/p)$ (cf. Section 2) and by F_δ the inhomogeneity obtained by substituting the solution $(\hat{u}^\delta, L_\delta)$ into the nonlinearity of the equation (as defined in Section 2), that is,

$$F_\delta = \frac{L'}{L}\langle y\hat{u}^\delta_y - \gamma_1'\gamma_1\hat{u}^\delta - \frac{1}{\epsilon}, \cdot\rangle \in C^\rho\big((0,\infty), W_p^{2s-2}(0,1)\big)\,,$$

for an existing $\rho \in (0,1)$ as follows from Theorem 12.1 in [3]. It follows from well-known spectral properties of the generator $-A$ that T_N admits the decomposition

$$T_N(t) = \begin{bmatrix} id & 0\\ 0 & \tilde{T}_N(t)\end{bmatrix}$$

on $PW_p^{2s-2} \oplus (1-P)W_p^{2s-2}$, where P is the continuous projection

$$Pv := \langle v, \mathbf{1}\rangle \;\big[= \int_0^1 v\,dx\big]\,,\;\; v \in W_p^{2s-2}$$

onto the kernel of the generator $-A_N$. By $\mathbf{1}$ we denoted the constant function with value 1 and the equality in brackets is only valid for $v \in L_p$. Furthermore the semigroup induced by T_N on $(1-P)E_0 := P_cE_0$ is exponentially decaying. The system of integral representations may thus be splitted as follows by using the decomposition of T_N:

$$\begin{aligned}P\hat{u}^\delta(t) &= P\hat{u}_0 + \int_0^t PF_\delta(\tau)\,d\tau\\ P_c\hat{u}^\delta(t) &= \tilde{T}_N\Big(\int_0^t \frac{\delta}{L_\delta(\sigma)}\,d\sigma\Big)\hat{u}_0 + \int_0^t \tilde{T}_N\Big(\int_\tau^t \frac{\delta}{L_\delta(\sigma)}\,d\sigma\Big)P_cF_\delta(\tau)\,d\tau\\ L_\delta(t) &= L_\delta + \int_0^t (\gamma_1\hat{u}^\delta(\tau))^n\,d\tau\,.\end{aligned}$$

From Lemmas 3.1 and 3.3 and the compact embedding

$$BUC^1\big([0,T]\big) \hookrightarrow BUC^\sigma\big([0,T]\big)$$

which is valid for any $T > 0$ we conclude that we may substract from L_δ a converging subsequence L_{δ_k} with

$$L_{\delta_k} \longrightarrow L_* \text{ in } BUC^\sigma\big([0,T]\big)\ (k \to \infty)\,. \tag{4.3}$$

The given space decomposition entails

$$A_N = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_N \end{bmatrix}. \quad \text{Setting} \quad \tilde{A}_N(t,\delta) = \frac{\delta}{L_\delta^2(t)} \tilde{A}_N$$

it thus follows that

$$\big[(\delta,t) \mapsto \delta^{-1}\tilde{A}_N(t,\delta)\big] \in C^\sigma\big([\delta_0,\infty] \times [0,T], \mathcal{H}^-(PE_1,PE_0)\big)$$

and that

$$\delta_k^{-1}\tilde{A}_N(t,\delta_k) \longrightarrow \frac{\tilde{A}_N}{L_*(t)}$$

as k tends to infinity. Observe now that

$$F_\delta(\tau) = \frac{L_\delta'(\tau)}{L_\delta(\tau)} \langle y\hat{u}_y^\delta(\tau) - \gamma_1'\gamma_1\hat{u}^\delta(\tau) - \epsilon^{-1}, \cdot\rangle \in W_p^{s-2+2\varepsilon}(0,1)$$

from which we infer, since $(\hat{u}^\delta, L_\delta)$ is a classical solution, that

$$PF_\delta(\tau) = \frac{L_\delta'(\tau)}{L_\delta(\tau)} \Big[\langle y\hat{u}_y^\delta(\tau), \mathbf{1}\rangle - \langle \gamma_1\hat{u}^\delta(\tau), 1\rangle - \epsilon\Big] = \frac{L_\delta'(\tau)}{L_\delta(\tau)} \big[P\hat{u}^\delta(\tau) - \epsilon^{-1}\big]\,.$$

Hereby we used that

$$P\langle y\hat{u}_y^\delta, \cdot\rangle = \int_0^1 y\hat{u}_y\,dy = \hat{u}^\delta(1,\tau) - P\hat{u}^\delta(\tau)$$

and that

$$P\langle \gamma_1'\gamma_1\hat{u}^\delta, \cdot\rangle = \hat{u}^\delta(1,\tau)\,.$$

Thus, on the other hand, we have

$$(1-P)F_\delta(\tau) = \frac{L_\delta'(\tau)}{L_\delta(\tau)} \big[\langle y\hat{u}_y^\delta(\tau) - \gamma_1'\gamma_1\hat{u}^\delta - \epsilon^{-1}, \cdot\rangle - \hat{u}^\delta(1,\tau) - P\hat{u}^\delta(\tau)\big]\,.$$

Using the convergence (4.3) and the bounds obtained in Lemmas 3.1 and 3.3 for u and u_x in $L_\infty \hookrightarrow L_p$ we see that

$$\frac{1}{\delta_k}(1-P)F_{\delta_k}(\tau) \longrightarrow 0 \qquad (k \to \infty)$$

uniformly with respect to $t \in [0,\infty)$. We are now in a position to apply Theorem A.5 of the appendix to conclude that

$$(1-P)\hat{u}^{\delta_k}(t) \longrightarrow 0 \qquad (k \to \infty)$$

uniformly on compact subsects of $(0,\infty)$. Observe then that the relation

$$\int_0^1 \hat{u}^\delta(t)\,dx = \frac{1}{L_\delta(t)} - -\frac{1}{\epsilon}$$

entails the convergence of $P\hat{u}^{\delta_k}$ in $BUC^\sigma(0,\infty)$ toward some u_*, since we have already established in Lemmas 3.1 and 3.3 that $\frac{\epsilon}{1+\epsilon} \le L_\delta(t) \le \epsilon$ and that $0 \le L'_\delta(t) \le 1$ for $t > 0$. Recall now that

$$P\hat{u}^\delta(t) = P\hat{u}_0 + \int_0^t PF_\delta(\tau)\,d\tau = 1 + \int_0^t \frac{L'_\delta(\tau)}{L_\delta(\tau)}\big[P\hat{u}^\delta - \frac{1}{\epsilon}\big]\,.$$

We conclude that

$$u_*(t) = 1 + \int_0^t \frac{L'_*(\tau)}{L_*(\tau)}\big(u_*(\tau) - \frac{1}{\epsilon}\big)\,.$$

Furthermore

$$\gamma_1\hat{u}^{\delta_k}(t) = \gamma_1(1-P)\hat{u}^{\delta_k}(t) + \gamma_1 P\hat{u}^{\delta_k}(t) \longrightarrow u_*(t)$$

uniformly on compact subintervals of $(0,\infty)$, which by (1.11) entails

$$L'_*(t) = u_*^n(t)\,.$$

Finally we can conclude that

$$\big(\hat{u}^{\delta_k}, L_{\delta_k}\big)(t) \longrightarrow (u_*, L_*)(t)$$

uniformly on compact subintervals of $(0,\infty)$ and (u_*, L_*) solves (4.1)–(4.2). Thus $(u_*, L_*) = (u_\infty, L_\infty)$. Assume now that the convergence only takes place for the subsequence L_{δ_k}. Then there exists a neighbourhood $\mathcal{U}$ of L_∞ in BUC^σ and a different subsequence $L_{\delta'_k}$ which has empty intersection with $\mathcal{U}$. The boundedness of the original net in C^1 implies that a further subsequence of $L_{\delta'_k}$ converges towards a limit in BUC^σ. The above arguments then show that this limit has to be L_∞ which is of course a contradiction. Thus we conclude that $L_\delta \to L_\infty$, which, in its turn, entails, as we showed above, that $Pu^\delta \to u_*$ and the proof is complete. □

Appendix: Singular perturbations

In this section we are concerned with singular perturbations of abstract evolution equations. We first need to fix some useful notation. Given two Banach spaces E_0 and E_1 with $E_1 \overset{d}{\hookrightarrow} E_0$ we write $\mathcal{H}^-(E_1, E_0) \subset \mathcal{H}(E_1, E_0)$ for the subset of all negative generators $-A$ of analytic, strongly continuous and exponentially decaying semigroups on E_0 with $\mathrm{dom}\,(A) \doteq E_1$ and endowe it with the induced topology of $\mathcal{L}(E_1, E_0)$. As in Section 2

$$E_\alpha = (E_0, E_1)_\alpha\,,\ \alpha \in (0,1)\,, p \in (0,\infty)$$

denotes the interpolation space obtained by the standard real interpolation functor. We also recall that $J = [0,T]$ for some $T \in (0,\infty)$ and that $\dot{J} = (0,T]$. Consider now the following Cauchy problem on J in the Banach space E_0:

$$\dot{u} + A(t,\varepsilon)u = f(t,\varepsilon)\,,\ \ u(0) = x \in E_0\,. \qquad (\mathrm{A.2})_\varepsilon$$

We are of course interested in the limit when ε tends to zero. The next proposition deals with the simple case when A is time independent.

Proposition A.1. *Assume that, for fixed $\varepsilon_0 > 0$,*

$$\big[\varepsilon \mapsto \varepsilon A(\varepsilon)\big] \in C\big([0,\varepsilon_0], \mathcal{H}^-(E_1,E_0)\big)\,,$$

that $\rho \in (0,1)$, and that $f_0 \in C^\rho(J,E_0)$ and $f : J \times (0,\varepsilon_0] \to E_0$. Let, moreover, A satisfy the following conditions:

$$\begin{aligned} &\text{(i) } f(\cdot,\varepsilon) \in C^\rho(J,E_0) \quad \textit{for } \varepsilon \in (0,\varepsilon_0]\,;\\ &\text{(ii) } \varepsilon f(\cdot,\varepsilon) \longrightarrow f_0 \ \textit{in } C(J,E_0) \ \textit{as } \varepsilon \to 0\,;\\ &\text{(iii) } \varepsilon A(\varepsilon) \longrightarrow A_0 \ \textit{in } \mathcal{L}(E_1,E_0) \ \textit{as } \varepsilon \to 0\,. \end{aligned}$$

Now let $u(\cdot,\varepsilon)$ be the unique solution of the Cauchy problem

$$\dot v + A(\varepsilon)v = f(t,\varepsilon)\ ,\ 0 < t \in J, \quad v(0) = x \in E_0$$

in E_0. Then for each $\alpha \in [0,1)$:

$$u(t,\varepsilon) \longrightarrow A_0^{-1} f_0(t) \ \textit{in } E_\alpha \ \textit{as } \varepsilon \to 0$$

uniformly with respect to $t \in \dot J$.

Proof. We know from the abstract theory of Cauchy Problems (cf. Paragraph II.2 in [4] for instance) that the solution of $(CP)_\varepsilon$ is given by the variation-of-constants-formula, that is,

$$u(t,\varepsilon) = e^{-tA(\varepsilon)}x + \int_0^t e^{(t-\tau)A(\varepsilon)} f(\tau,\varepsilon)\, d\tau. \tag{A.3}$$

On the other hand it is known from semigroup theory (cf. e. g. [10] and [7]) that the resolvent of a generator $-A$ is given by the Laplace transform of the generated semigroup T_A. This means that, for $\mathrm{Re}(\lambda) > \mathrm{type}(-A)$

$$(\lambda + A)^{-1}x = \int_0^\infty e^{-\lambda\sigma} T_A(\sigma)x\, d\sigma\ ,\ x \in E_0\,.$$

In our case we can choose $\lambda = 0$ and obtain for fixed $t > 0$

$$A_0^{-1} f_0(t) = \int_0^\infty T_{A_0}(\sigma) f_0(t)\, d\sigma = \int_0^\infty e^{-\sigma A_0} f_0(t)\, d\sigma.$$

Observe now that we can rewrite (A.3) as follows

$$u(t,\varepsilon) = e^{-tA(\varepsilon)}x + \int_0^{\frac{t}{\varepsilon}} e^{-\sigma\varepsilon A(\varepsilon)} \varepsilon f(t-\varepsilon\sigma,\varepsilon)\, d\sigma,$$

performing the elementary change of variables $\sigma := \frac{t-\tau}{\varepsilon}$. Thus we have to prove that

$$\int_0^\infty e^{-\sigma A_0} f_0(t)\, d\sigma - \int_0^{\frac{t}{\varepsilon}} e^{-\sigma\varepsilon A(\varepsilon)} \varepsilon f(t-\varepsilon\sigma,\varepsilon)\, d\sigma - e^{-tA(\varepsilon)}x \stackrel{\varepsilon\to 0}{\longrightarrow} 0$$

in the topology of E_α. The above left-hand side may also be written in the following manner:

$$\int_0^\infty \big(e^{-\sigma A_0} - e^{-\sigma\varepsilon A(\varepsilon)}\big) f_0(t)\, d\sigma + \int_{\frac{t}{\varepsilon}}^\infty e^{-\sigma\varepsilon A(\varepsilon)} f_0(t)\, d\sigma +$$
$$\int_0^{\frac{t}{\varepsilon}} e^{-\sigma\varepsilon A(\varepsilon)} \big(f_0(t) - f_0(t-\varepsilon\sigma)\big)\, d\sigma +$$
$$\int_0^{\frac{t}{\varepsilon}} e^{-\sigma\varepsilon A(\varepsilon)} \big(f_0(t-\varepsilon\sigma) - \varepsilon f(t-\varepsilon\sigma,\varepsilon)\big)\, d\sigma - e^{-\frac{t}{\varepsilon}\varepsilon A(\varepsilon)}x =: \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}\ .$$

The first term may be rewritten as

$$\mathrm{I} = \big\{(A_0)^{-1} - (\varepsilon A(\varepsilon))^{-1}\big\} f_0(t)\,.$$

Thus its convergence is implied by the assumptions. The remaining terms are easily seen to vanish for $\varepsilon \to 0$ using the assumptions on A, f and f_0. Term V prevents the convergence from being uniform up to the initial time. □

We now turn to the more involved case of a generating family which is time dependent. In order to be able to prove a similar result as in Proposition A.1 we need two lemmas.

Lemma A.2. *Let A be such that*

$$\big[(t,\varepsilon) \mapsto \varepsilon A(t,\varepsilon)\big] \in C^\rho\big(J\times[0,\varepsilon_0], \mathcal{H}^-(E_1,E_0)\big)$$

for some $\rho \in (0,1)$ and $\varepsilon_0 > 0$. Denote by U_ϵ the evolution operator generated by $A(\cdot,\varepsilon)$. Then, for $\alpha \in [0,1)$ and $t > 0$,

$$\|U_\varepsilon(t,\tau)\|_{0\to\alpha} \le c\, \frac{\varepsilon^\alpha}{(t-\tau)^\alpha} e^{-\omega\frac{t-\tau}{\varepsilon}}$$

provided $\tau \in [0,t)$.

Proof. We recall (cf. [4], II.2.2) that the evolution operator satisfies the following weakly singular Volterra integral equation:

$$U_A(t,\tau) = e^{-(t-\tau)A(\tau)} - \int_\tau^t U_A(t,\sigma)[A(\sigma)-A(\tau)]e^{-(\sigma-\tau)A(\tau)}\, d\sigma\,, \tag{A.4}$$

If we set

$$a(t,\tau)=e^{-(t-\tau)A(\tau)}\,,\ h(t,\tau)=[A(\tau)-A(t)]e^{-(t-\tau)A(\tau)}$$

and

$$w=\sum_{n=1}^{\infty}\underbrace{h*\ldots*h}_{n\text{ times}}\,,$$

it is well known (cf. Theorem II.3.2.2 in [4]) that the unique solution u of the above integral equation is given by

$$u=a+a*w\,,$$

where

$$a*w(t,\tau):=\int_{\tau}^{t}a(t,\sigma)w(\sigma,\tau)\,d\sigma\,.$$

Thus, setting

$$a_\varepsilon(t,\tau):=e^{-(t-\tau)A(\tau,\varepsilon)}\,,\ h_\varepsilon:=[A(\tau,\varepsilon)-A(t,\varepsilon)]a_\varepsilon(t,\tau)$$

and

$$w_\varepsilon:=\sum_{n=1}^{\infty}\underbrace{h_\varepsilon*\ldots*h_\varepsilon}_{n\text{ times}}\,,$$

we obtain

$$U_\varepsilon(t,\tau)=a_\varepsilon(t,\tau)+a_\varepsilon*w_\varepsilon(t,\tau)=:\mathrm{I}_\varepsilon+\mathrm{II}_\varepsilon\,.$$

It is easily seen that the assumptions imply

$$\|\mathrm{I}_\varepsilon\|_{\beta\to\alpha}=\big\|e^{-\frac{t-\tau}{\varepsilon}\varepsilon A(\tau,\varepsilon)}\big\|_{\beta\to\alpha}\le c\,\frac{\varepsilon^{\alpha-\beta}}{(t-\tau)^{\alpha-\beta}}e^{-\omega\frac{t-\tau}{\varepsilon}}\,.$$

As for the second term we first observe that

$$\begin{aligned}\|h_\varepsilon\|_{0\to0}&\le\varepsilon^{-1}\big\|\varepsilon A(t,\varepsilon)-\varepsilon A(\tau,\varepsilon)\big\|_{1\to0}\big\|e^{-\frac{t-\tau}{\varepsilon}\varepsilon A(\tau,\varepsilon)}\big\|_{0\to1}\\&\le\varepsilon^{-1}|t-\tau|^{\rho}\frac{\varepsilon}{t-\tau}e^{-\omega\frac{t-\tau}{\varepsilon}}=c\,|t-\tau|^{\rho-1}e^{-\omega\frac{t-\tau}{\varepsilon}}\,.\end{aligned}$$

We then inductively obtain for $h_\varepsilon^{*n}:=\underbrace{h_\varepsilon*\ldots*h_\varepsilon}_{n\text{ times}}$

$$\|h_\varepsilon^{*n}(t,\tau)\|_{\beta\to0}\le c^ne^{-\omega\frac{t-\tau}{\varepsilon}}(t-\tau)^{n\rho-1}\frac{\Gamma(\rho)^n}{\Gamma(n\rho)}\,,$$

which leads to

$$\|w_\varepsilon(t,\tau)\|_{\beta\to0}\le c\,e^{-\omega\frac{t-\tau}{\varepsilon}}$$

and to

$$\mathrm{II}_\varepsilon \le c\,\frac{\varepsilon^\alpha}{(t-\tau)^\alpha} e^{-\omega\frac{t-\tau}{\varepsilon}}\,.$$

Thus we finally see that

$$\|U_\varepsilon(t,\tau)\|_{0\to\alpha} \le c\,\frac{\varepsilon^\alpha}{(t-\tau)^\alpha} e^{-\omega\frac{t-\tau}{\varepsilon}}\,.$$

The assertion now easily follows. We observe that the given proof follow the steps of the proof of Lemma 1.6 in [9] with only slight modifications. We thus refrained from giving all the details. □

Lemma A.3. *Assume that*

$$\big[(t,\varepsilon)\mapsto \varepsilon A(t,\varepsilon)\big] \in C^\rho\big(J\times[0,\varepsilon_0],\mathcal{H}^-(E_1,E_0)\big)$$

for some $\rho\in(0,1)$ *and* $\varepsilon_0>0$ *and that* $f\in C^\rho(J,E_0)$ *. Let again* U_ε *denote the evolution operator generated by* $A(\cdot,\varepsilon)$ *and* $\alpha\in(0,1)$ *. Then*

$$\frac{1}{\varepsilon}\int_0^t \big(U_\varepsilon(t,\tau)-e^{-A(t,\varepsilon)(t-\tau)}\big) f(\tau)\,d\tau \;\longrightarrow 0\,,\ \varepsilon\to 0\,,$$

in the topology of E_α *.*

Proof. Observe that U_ε also satisfies the following Volterra equation equivalent to (A.4):

$$U_\varepsilon(t,\tau) = e^{-(t-\tau)A(t,\varepsilon)} - \int_\tau^t e^{-(t-\sigma)A(t,\varepsilon)}[A(t,\varepsilon)-A(\sigma,\varepsilon)]U_\varepsilon(\sigma,\tau)\,d\sigma\,.$$

It follows by Theorem II.3.2.2 in [4] that

$$U_\varepsilon(t,\tau) - e^{-A(t,\varepsilon)(t-\tau)} = \int_\tau^t w^\varepsilon(t,\sigma)e^{-A(\tau,\varepsilon)(\sigma-\tau)}\,d\sigma$$

for

$$w^\varepsilon(t,\tau) := \sum_{n=1}^\infty \underbrace{h^\varepsilon * \dots * h^\varepsilon}_{n \text{ times}}$$

and

$$h^\varepsilon(t,\tau) := e^{-(t-\tau)A(t,\varepsilon)}[A(t,\varepsilon)-A(\tau,\varepsilon)]\,.$$

Thus the estimate

$$\begin{aligned}\big\|\varepsilon^{-1}U_\varepsilon(t,\tau)-\varepsilon^{-1}e^{-A(t,\varepsilon)(t-\tau)}\big\|_{0\to\alpha} &\le \frac{c}{\varepsilon}\int_\tau^t \frac{\varepsilon^\alpha}{(\sigma-\tau)^\alpha}e^{-\omega\frac{t-\sigma}{\varepsilon}}e^{-\omega\frac{\sigma-\tau}{\varepsilon}}\,d\sigma\\ &\le c\varepsilon^{\alpha-1}e^{-\omega\frac{t-\tau}{\varepsilon}} \le c\frac{\varepsilon^{\alpha-\delta}}{(t-\tau)^{1-\delta}}\end{aligned}$$

is valid for each $\delta \in (0, \alpha)$, since it is easily seen that for w^ε the same estimates hold as for w_ε. The assertion now follows from

$$\begin{aligned}\Big\|\frac{1}{\varepsilon}\int_0^t \big(U_\varepsilon(t,\tau) - e^{-A(t,\varepsilon)(t-\tau)}\big) f(\tau)\, d\tau\Big\|_\alpha &\le c\int_0^t \frac{\varepsilon^{\alpha-\delta}}{(t-\tau)^{1-\delta}} \|f(\tau)\|_0\, d\tau \\ &\le c\,\delta^{-1}\|f\|_{C(J,E_0)}\varepsilon^{\alpha-\delta} t^\delta\,.\end{aligned}$$

□

Remark A.4. It is easily seen by the last estimate in the above proof that the convergence stated in the preceding lemma is uniform on bounded subsets of $C(J, E_0)$ for f.

We are now ready for the following

Theorem A.5. *Let the assumptions of Proposition* A.1 *be satisfied and assume, in addition, that* $\big[(t,\varepsilon) \mapsto \varepsilon A(t,\varepsilon)\big] \in C^\rho\big(J \times [0,\varepsilon_0], \mathcal{H}^-(E_1, E_0)\big)$ *for some* $\rho \in (0,1)$ *and given* $\varepsilon_0 > 0$. *Suppose that*

$$\varepsilon A(t,\varepsilon) \longrightarrow A_0(t) \ \textit{as}\ \varepsilon \to 0$$

in $\mathcal{L}(E_1, E_0)$ *on* J. *Denote by* $u(\cdot,\varepsilon)$ *the unique solution of the Cauchy problem* $(\mathrm{A.2})_\varepsilon$ *in* E_0 *with* $u_0 \in E_0$. *Then:*

$$u(t,\varepsilon) \longrightarrow A_0(t)^{-1} f_0(t) \ \textit{in}\ E_\alpha \ \textit{as}\ \varepsilon \to 0$$

uniformly with respect to t *in compact subintervals of* $\dot{J}$.

Proof. Let $t > 0$ be fixed. The value $u(t,\varepsilon)$ of the solution at time t is given by

$$\begin{aligned}u(t,\varepsilon) &= U_\varepsilon(t,0)x + \int_0^t U_\varepsilon(t,\tau) f(\tau,\varepsilon)\, d\tau \\ &= U_\varepsilon(t,0)x + \int_0^t \big\{U_\varepsilon(t,\tau) - e^{-(t-\tau)A(t,\varepsilon)}\big\} f(\tau,\varepsilon)\, d\tau \\ &\qquad + \int_0^t e^{-(t-\tau)A(t,\varepsilon)} f(\tau,\varepsilon)\, d\tau =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{III}_\varepsilon\,.\end{aligned}$$

Lemmas A.2 and A.3 imply that

$$\mathrm{I}_\varepsilon\,,\ \mathrm{II}_\varepsilon \longrightarrow 0 \text{ in } E_\alpha\,, \text{ as } \varepsilon \to 0\,.$$

On the other hand we can apply Proposition A.1 to obtain

$$\mathrm{III}_\varepsilon \longrightarrow A_0(t)^{-1} f_0(t) \text{ in } E_\alpha\,, \text{ as } \varepsilon \to 0\,.$$

The Theorem is thus proved. □

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References

[1] T. ALFREY, E.F. GURNEE, W.G. LLOYD. Diffusion in glassy polymers. *J. Polymer Sci. Part C*, 12:249–261, 1996.

[2] H. AMANN. *Ordinary Differential Equations*. de Gruyter, Berlin, 1990.

[3] H. AMANN. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. *Function Spaces, Differential Operators and Nonlinear Analysis, Teubner-Texte zur Math.*, 133:9–126, 1993.

[4] H. AMANN. *Linear and Quasilinear Parabolic Problems*. Birkhäuser, Basel, 1995.

[5] G. ASTARITA, G.C. SARTI. A class of mathematical models for sorption of swelling solvents in glassy polymers. *Polymer Engng. and Sci.*, 18:388–395, 1978.

[6] D.S. COHEN, T. ERNEUX. Free boundary problems in controlled release pharmaceuticals. I. Diffusion in glassy polymers. *SIAM J. Appl. Math.*, 48:1451–1465, 1988.

[7] J.A. GOLDSTEIN. *Semigroups of Linear Operators and Applications*. Oxford University Press, Oxford, 1985.

[8] P. GUIDOTTI. Diffusion in glassy polymers: A free boundary problem. *Adv. Math. Sci. Appl.*, 7.

[9] P. GUIDOTTI. Singular quasilinear abstract Cauchy problems. *Nonlin. Anal. TMA*, 32(5):667–695, 1998.

[10] A. PAZY. *Semigroups of Linear Operators and Application to Partial Differential Equations*. Springer Verlag, New York, 1983.

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