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SINGULAR QUASILINEAR ABSTRACT CAUCHY PROBLEMS

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1. INTRODUCTION

The aim of this paper is the study of abstract Cauchy problems which are singular in the sense we are to explain. Let $J = [0, T]$ for some $T > 0$ and $\dot{J} = (0, T]$. Consider the Cauchy problem

$$\dot{u} + A(t)u = f(t), \qquad t \in \dot{J} \tag{0.1}$$

in a given Banach space E_0, where, for $t > 0$, the operators $-A(t)$ are assumed to generate analytic C_0-semigroups with a constant common domain of definition satisfying

$$(\mathrm{dom}(A(t)), \|\cdot\|_{A(t)}) \doteq E_1 \tag{0.2}$$

for a given Banach space $E_1 \hookrightarrow E_0$. Suppose now that the family of generators A behaves like t^{-k} ($k \in (1, \infty)$) as t tends to 0 in the topology of $\mathcal{L}(E_1, E_0)$, that is, that there exists a generator A_0 such that

$$t^k A(t) \to A_0 \quad (t \to 0) \qquad \text{in } \mathcal{L}(E_1, E_0).$$

Then we are interested in finding conditions about the data for which the Cauchy problem (0.1) is solvable. It is well known (cf. the original works [1] and [2] or the monograph [3]) that if the family A satisfies the conditions

(i) $-A(t)$ generates an analytic C_0-semigroup for each fixed $t \geq 0$ and condition (0.2) is satisfied,

(ii) $A \in C^\rho(J, \mathcal{L}(E_1, E_0))$

then there exists an evolution operator $U = \{U(t, \tau) \mid 0 \leq \tau \leq t \leq T\}$ which characterizes the solutions of the following regular Cauchy problem

$$\dot{u} + A(t)u = f(t), \quad t \in \dot{J}, \qquad u(0) = x. \tag{0.3}$$

Of course conditions concerning the data have to be imposed in order to obtain existence of a solution. But if a solution exists, then it is given by the so-called variation-of-constants-formula, that is by

$$u(t) = U(t, 0)x + \int_0^t U(t, \tau) f(\tau)\, d\tau. \tag{0.4}$$

Given a singular family of generators A and $\delta > 0$, we may now assume that condition (ii) above is satisfied on $[\delta, T]$. In this case, fixing an arbitrary initial datum u_δ, we can solve (0.1) in $(\delta, T]$. Then we know that the solution $u(\cdot, \delta)$ is given by the variation-of-constants-formula

$$u(t, \delta) = U_\delta(t, \delta)u_\delta + \int_\delta^t U_\delta(t, \tau)f(\tau)\,d\tau, \tag{0.5}$$

where by U_δ we denoted the evolution operator generated by $A|_{[\delta, T]}$. If we suppose, in addition, that the semigroups generated by the operators $-A(t)$, for $t \in \dot{J}$, and A_0 are exponentially decaying at a common rate, then we are able to introduce a singular evolution operator for (0.1). The analysis of his properties will allow us to prove the validity of a singular variation-of-constants-formula for the singular Cauchy problem. Formally the formula may be obtained by taking the limit in (0.5) on the assumption that U_δ converges to some U_A, in some reasonable topology, as δ tends to 0. In this case the limiting formula would indeed provide us with a candidate for the singular variation-of-constants-formula and U_A with one for the singular evolution operator. This formal idea can be carried out in a rigorous way and leads to the following main proposition.

Main proposition. Assume that a family of operators A and a function f are given which satisfy the following properties:

(i) $-A(t)$ generates, for each $t \in \dot{J}$, a decaying (at a given fixed rate) analytic C_0-semigroup,

(ii) $[t \mapsto t^k A(t)] \in C^\rho(J, \mathcal{L}(E_1, E_0))$ for some $\rho \in (0, 1)$,

(iii) $f \in L_{1,\mathrm{loc}}(J, E_0)$.

Then there exists a unique singular evolution operator U_A which characterizes the solution of the singular Cauchy problem (0.1). In other words: if $u\colon J \to E_0$ is a solution of (0.1), it is necessarily given by the singular variation-of-constants-formula for the singular evolution operator U_A, that is by

$$u(t) = \int_0^t U_A(t, \tau)f(\tau)\,d\tau.$$

In the next section we define the singular evolution operator and show it has all the properties we need to prove the main proposition. The proof rests on careful estimates for the singular propagator. The further analysis is based on the representation formula. Indeed it will be by a fixed point argument for the singular variation-of-constants-formula that the following main theorem concerning quasilinear singular Cauchy problems is proved in the last section. We renounce, at this point, to give the precise hypotheses concerning the data needed to prove the theorem.

Main theorem. Let, for each $\theta \in (0, 1)$, E_θ be a suitable interpolation space between E_0 and E_1. Let $0 < \beta \leq \alpha < 1$ and assume that X_α is an open subset of E_α containing 0. Suppose further that

$$A\colon J \times X_\alpha \to \mathcal{L}(E_1, E_0), \qquad (t, u) \mapsto A(t, u)$$

and

$$f\colon J \times X_\alpha \to E_\beta, \qquad (t, u) \mapsto F(t, u)$$

satisfy appropriate hypotheses. Then the quasilinear Cauchy problem

$$\dot{u} + A(t, u)u = F(t, u), \qquad t \in \dot{J} \tag{0.6}$$

possesses a unique local solution u on $[0, T_+]$, for some $T_+ > 0$, with $u \in C([0, T_+], X_\alpha)$. It is the unique fixed point of the singular variation-of-constants-formula

$$u(t) = \int_0^t U_{A(\cdot,u)}(t, \tau)F(\tau, u(\tau))\, d\tau.$$

Moreover the solution u enjoys the following regularity properties

$$C^\delta([0, T], E_\varepsilon) \cap C((0, T], E_1) \cap C^1((0, T], E_0),$$

provided that $\varepsilon, \delta \in (0, 1)$ satisfy $\varepsilon \geq 1 - 1/k$ and $\delta < \varepsilon \wedge (1 - \varepsilon)$.

In the concluding section we turn our interest to singular initial boundary value problems, which can be solved using the abstract result of the previous sections.

Remark 0.1. Let us say a word concerning the reasons why the above introduced class of singular evolutionary equations deserve some attention. When studying free boundary problems characterized by the onset of a phase one is naturally led to singular Cauchy problems by trying to transform the problem to an initial value problem in a fixed domain. As a consequence of the topological change of the domain this may namely only be done at the expense of allowing a singular coordinate transformation which is, in its turn, reflected in the singular behaviour of the obtained Cauchy problem. For a concrete application to a free boundary problem possessing the sketched features we refer to [4].

Remark 0.2. A wider class of singular Cauchy problems has already been studied in the literature. We mention in particular the works of Favini (cf. [5, 6] and the references therein), where the author (and co-authors) discuss existence and regularity of a broad class of singular Cauchy problems, not necessarily of parabolic type, but always linear or semilinear. However, the approach we shall present in this paper seems to be new and applies to more general situations in the sense explained below. On the one hand, it has the advantage of relying on an "explicit" formula for the solution (cf. the main proposition). On the other hand it provides a natural way of studying the quasilinear version of $(1.3)_{(A,f)}$, which has not been considered in the literature so far. Lastly it allows good estimates about the dependence of the solution on the singular behaviour of family of generators. This turns out to be of paramount importance in concrete applications (cf. [4]).

1. THE SINGULAR EVOLUTION OPERATOR

We first specify the class of singular Cauchy problems, in which we are interested and fix the notation which we shall need throughout the paper.

Definition. Let $J \subset \mathbb{R}^+$ be a nontrivial interval containing the origin and set $\dot{J} := J \setminus \{0\}$. Assume also that E is a normed vector space and F a subset of E. Then, for given $m \in \mathbb{R}^+$ and $k \in (1, \infty)$, we put

$$C_k^m(J, F) := \{f \in C^m(\dot{J}, F) \mid t \mapsto t^k f(t) \in C^m(J, F)\}.$$

If F is a subspace of E then $C_k^m(J, F)$ becomes a normed vector space, if endowed with the norm

$$\|f\|_{C_k^m} := \|t^k f(t)\|_{C^m}, \qquad f \in C_k^m(J, E).$$

If E is complete and F a closed subspace then $C_k^m(J, F)$ becomes a Banach space with respect to the above norm. It may be viewed as the space "$(1/t^k)C^m(J, F)$".

Let now E_0 and E_1 be Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$. Then $\mathcal{H}(E_1, E_0)$ denotes the set of negative generators A of analytic C_0-semigroups on E_0 with

$$(\mathrm{dom}(A), \|\cdot\|_A) \doteq E_1,$$

where $\|\cdot\|_A$ stands for the graph norm on $\mathrm{dom}(A)$. Lastly the subset of $\mathcal{H}(E_1, E_0)$ consisting of the negative generators of exponentially decaying semigroups is denoted by $\mathcal{H}^-(E_1, E_0)$. Let $k > 1$. We assume that

$$A \in C_k^\rho(J, \mathcal{H}^-(E_1, E_0)) \tag{1.1}$$

for some $\rho \in (0, 1)$ and that

$$f \in C_{k-1}(J, E_0). \tag{1.2}$$

Remark 1.1. Let $m, l \in (1, \infty)$. We observe that $C_m(J, E_0) \subset C_l(J, E_0)$ for $m \leq l$, whereas

$$C_m^\rho(J, \mathcal{H}^-(E_1, E_0)) \cap C_l^\rho(J, \mathcal{H}^-(E_1, E_0)) = \varnothing$$

for $m \neq l$ in general. In fact, it follows from $l > m$ and from

$$A \in C_m^\rho(J, \mathcal{H}^-(E_1, E_0))$$

that $\lim_{t\to 0} t^l A(t) = 0$. Thus, if $E_1 \neq E_0$,

$$A \notin C_l^\rho(J, \mathcal{H}^-(E_1, E_0)),$$

since $0 \notin \mathcal{H}^-(E_1, E_0)$.

Definition. Let hypotheses (1.1) and (1.2) be satisfied. Then we call

$$\dot{u} + A(t)u = f(t), \qquad t \in \dot{J}, \tag*{$(1.3)_{(A,f)}$}$$

a singular Cauchy problem of parabolic type.

Remark 1.2. It will turn out that a continuous solution of $(1.3)_{(A,f)}$ necessarily takes on the initial value 0. This explains why we do not prescribe any initial datum, but still call $(1.3)_{(A,f)}$ a Cauchy problem.

In order to find solutions of singular Cauchy problems we shall construct a "singular evolution operator"

$$U_A := \{U_A(t, \tau) \mid 0 \leq \tau \leq t \in J\}$$

for the singular family A, which characterizes the solution of $(1.3)_{(A,f)}$.

Terminology. From now on we shall refer to evolution operators as regular evolution operators to distinguish them from the singular ones, which we are about to introduce. The same terminology will be used in connection with the corresponding generating families.

It is known that in the regular case the evolution operator U_A corresponding to a given family A may be obtained as the solution of a weakly singular Volterra integral equation. In fact U_A solves the following equation

$$U_A(t,\tau) = \mathrm{e}^{-(t-\tau)A(\tau)} - \int_\tau^t U_A(t,\sigma)[A(\sigma) - A(\tau)]\,\mathrm{e}^{-(\sigma-\tau)A(\tau)}\,\mathrm{d}\sigma, \tag{1.4}$$

or the equivalent

$$U_A(t,\tau) = \mathrm{e}^{-(t-\tau)A(\tau)} - \int_\tau^t \mathrm{e}^{-(t-\sigma)A(t)}[A(t) - A(\sigma)]U_A(\sigma,\tau)\,\mathrm{d}\sigma, \tag{1.5}$$

in a suitably chosen Banach space of operators. Here we denoted by

$$\{\mathrm{e}^{-\tau A(t)} \mid \tau \geq 0\}$$

the semigroup generated by $-A(t)$. It turns out that equations (1.4) and (1.5) are uniquely solvable in that space. The following almost trivial remark will be useful later.

Remark 1.3. Let a regular family $\{A(t) \mid t \in J\}$ be given. Then the value $U_A(t,\tau)$ of the generated evolution operator U_A only depends on the values of A in the interval $[\tau, t]$.

Proof. Consider the family $\{A(\sigma) \mid \sigma \in [\tau, t]\}$. It of course generates a regular evolution operator $\tilde{U}_A$ on $[\tau, t]$. By uniqueness we infer from (1.4) or (1.5) that

$$U_A(s,\sigma) = \tilde{U}_A(s,\sigma), \qquad \tau \leq \sigma \leq s \leq t. \quad \blacksquare$$

Since in many proofs we need to go back to the defining integral equation (1.4) for the evolution operator we fix some useful notation. We put

$$\begin{aligned} u(t,\tau) &:= U_A(t,\tau),\ a(t,\tau) := \mathrm{e}^{-(t-\tau)A(\tau)}, \\ h(t,\tau) &:= [A(\tau) - A(t)]\,\mathrm{e}^{-(t-\tau)A(\tau)}. \end{aligned} \tag{1.6}$$

Thus, setting

$$u * h(t,\tau) := \int_\tau^t u(t,\sigma)h(\sigma,\tau)\,\mathrm{d}\sigma,$$

we may rewrite (1.4) as

$$u = a + u * h \tag{1.7}$$

in a suitably chosen Banach space of operators. Solving (1.7) amounts to establishing the existence (in the chosen Banach space) of

$$w := \sum_{n=1}^{\infty} \underbrace{h * \cdots * h}_{n\ \text{times}} \tag{1.8}$$

In other words, only the convergence of the series in the topology of the chosen space has to be established. This is due to the fact that a solution of (1.7) is formally given by

$$u = a + a * w. \tag{1.9}$$

We now give a preliminary incomplete definition of the singular propagator.

Definition. Let A be a singular family satisfying (1.1) and define, for $\delta > 0$,

$$\{U_\delta(t, \tau) \mid \delta \leq \tau \leq t \in J\}$$

to be the propagator generated by $\{A(t) \mid t \in J \cap [\delta, \infty)\}$. It then follows from Remark 1.3 that

$$U_\delta(t, \tau) = U_{\delta'}(t, \tau), \qquad \delta \vee \delta' \leq \tau \leq t \in J.$$

for strictly δ and δ', and thus it is meaningful to put

$$U_A(t, \tau) := U_\delta(t, \tau)$$

for $0 < \tau \leq t \in J$ and, of course, $\delta \leq \tau$.

In the next proposition we establish a fundamental property of U_A, which will allow us to complete the above definition for $\tau = 0$ to obtain the singular evolution operator and which will also turn out to be very useful later. We now fix for the remainder of the paper a family of interpolation functors

$$\mathfrak{F} := \{(\cdot, \cdot)_\theta \mid \theta \in (0, 1)\}$$

with $(\cdot,\cdot)_\theta \in \{(\cdot, \cdot)_{\theta,p}, [\cdot, \cdot]_\theta \mid p \in [1, \infty)\}$ for $\theta \in (0, 1)$. Here we denoted by $(\cdot, \cdot)_{\theta,p}$ and by $[\cdot, \cdot]_\theta$ the standard real and complex interpolation functors, respectively. Then, given a pair (E_0, E_1) of Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$ (from now on referred to as densely injected Banach couple), we define intermediate spaces by means of the just introduced family of functors

$$E_\alpha := (E_0, E_1)_\alpha.$$

Observe also that we always set $(E_0, E_1)_j := E_j$ for $j = 0, 1$.

Remark 1.4. From now on we shall assume, without loss of generality, that the interval J is compact. As long as we are not interested in questions concerning global existence this is allowed.

To prove the next proposition we need two lemmas.

Notation. Let $-A$: $\mathrm{dom}(A) \subset E \to E$ be the generator of a C_0-semigroup T. If $M \geq 1$ and $\omega \in \mathbb{R}$ are constants for which the estimate

$$\|T(t)\|_{\mathcal{L}(E)} \leq M\,\mathrm{e}^{\omega t}$$

is valid, then we write $A \in \mathcal{G}(E, M, \omega)$.

LEMMA 1.5. Suppose that assumption (1.1) is met for the singular family A. Then there exist constants $M \geq 1$ and $\omega_0 > 0$ such that

$$t^k A(t) \in \mathcal{G}(E_j, M, -\omega_0), \qquad t \in J, j = 0, 1.$$

There exists, moreover, a neighbourhood $\mathfrak{A}$ of $\{t^k A(t) \mid t \in J\}$ in the topology of $\mathcal{L}(E_1, E_0)$ in which the above property remains valid.

Proof. The assertion is a consequence of the compactness of J, the continuity of $[t \mapsto t^k A(t)]$, the Cauchy-Dunford integral representation-formula for the semigroup and Theorem I.1.3.1 in [3]. ■

LEMMA 1.6. Let (E_0, E_1) be a densely injected Banach couple, $\mathfrak{F}$ be the above introduced family of interpolation functors and A be a singular family of generators satisfying (1.1). Assume further that $0 \leq \beta \leq \alpha \leq 1$ and that $\beta < \rho \wedge 1/k'$, k' being the dual exponent of k. Then there exists a constant $c > 0$ with

$$\|U_A(t, \tau)\|_{\beta \to \alpha} \leq c \frac{\tau^{k(\alpha-\beta)}}{(t - \tau)^{\alpha-\beta}}, \qquad t \in \dot{J}, \tau \in (0, t].$$

Proof. We use the notation introduced in (1.6). In [3] it is proved that

$$U_A = a + a * w. \tag{1.10}$$

We estimate the terms implicitly present in (1.10) separately. Let $0 < \tau \leq t \leq T$.

(α) The first term is easily estimated by means of semigroup and interpolation theory as follows:

$$\|a(t, \tau)\|_{\beta \to \alpha} = \|e^{-((t-\tau)/\tau^k)\tau^k A(\tau)}\|_{\beta \to \alpha} \leq c \frac{\tau^{k(\alpha-\beta)}}{(t - \tau)^{\alpha-\beta}} \exp\left(-\omega_0 \frac{t - \tau}{\tau^k}\right). \tag{1.11}$$

(β) To be able to estimate w we first need to establish a good estimate for h. We proceed in the following way:

$$\begin{aligned}\|h(t, \tau)\|_{0 \to 0} &\leq \|t^k A(t)(t^{-k} - \tau^{-k}) + \tau^{-k}(t^k A(t) - \tau^k A(\tau))\|_{1 \to 0} \| e^{-(t-\tau)A(\tau)}\|_{0 \to 1} \\ &\leq c\left\{\frac{\sup\|t^k A(t)\|_{1 \to 0}}{t^k \tau^k} k(t - \tau)t^{k-1} + c\frac{(t - \tau)^\rho}{\tau^k}\right\} \frac{\tau^k}{(t - \tau)} \exp\left(-\omega_0 \frac{t - \tau}{\tau^k}\right) \\ &\leq c[t^{-1} + (t - \tau)^{\rho-1}] \exp\left(-\omega_0 \frac{t - \tau}{\tau^k}\right). \end{aligned} \tag{1.12}$$

Since, on the one hand, the estimate

$$\frac{(t - \tau)^{1-\tilde{\rho}}}{\tau} \exp\left(-\omega_0 \frac{t - \tau}{\tau^k}\right) \leq c \exp\left(-\frac{\omega_0}{2} \frac{t - \tau}{\tau^k}\right), \tag{1.13}$$

is valid for $\tilde{\rho} := \rho \wedge 1/k'$ and

$$\left(\frac{t}{\tau}\right)^q \exp\left(-\omega_0 \frac{t - \tau}{\tau^k}\right) \leq c \exp\left(-\frac{\omega_0}{2} \frac{t - \tau}{\tau^k}\right), \tag{1.14}$$

for $q > 0$, on the other hand, we obtain that

$$\begin{aligned}
\frac{1}{t}\exp\left(-\omega_0\frac{t-\tau}{\tau^k}\right) &\le \frac{1}{\tau}\exp\left(-\omega_0\frac{t-\tau}{\tau^k}\right)\\
&= (t-\tau)^{\tilde{\rho}-1}\frac{(t-\tau)^{1-\tilde{\rho}}}{\tau}\exp\left(-\omega_0\frac{t-\tau}{\tau^k}\right)\\
&\le c(t-\tau)^{\tilde{\rho}-1}\exp\left(-\frac{\omega_0}{2}\frac{t-\tau}{\tau^k}\right)\\
&\le c(t-\tau)^{\tilde{\rho}-1}\frac{\tau^{k(\alpha-\beta)}}{t^{k(\alpha-\beta)}}\exp\left(-\frac{\omega_0}{4}\frac{t-\tau}{\tau^k}\right)\\
&\le c\frac{\tau^{k(\alpha-\beta)}}{t^{k(\alpha-\beta)}}(t-\tau)^{\tilde{\rho}-1}.
\end{aligned}$$

Thus we conclude that (1.12) can be further estimated arriving at

$$\frac{1}{c}\|h(t,\tau)\|_{\beta\to 0} \le \|h(t,\tau)\|_{0\to 0} \le c\frac{\tau^{k(\alpha-\beta)}}{t^{k(\alpha-\beta)}}(t-\tau)^{\tilde{\rho}-1}.$$

(γ) Next we turn to $h^{*n} := \underbrace{h * \cdots * h}_{n\text{ times}}$. Let us first estimate h^{*2}.

$$\begin{aligned}
\|h * h(t,\tau)|_{\beta\to 0} &\le \int_\tau^t \|h(t,\sigma)\|_{0\to 0}\|h(\sigma,\tau)\|_{\beta\to 0}\,\mathrm{d}\sigma\\
&\le c^2\int_\tau^t \frac{\sigma^{k(\alpha-\beta)}}{t^{k(\alpha-\beta)}}(t-\sigma)^{\tilde{\rho}-1}\frac{\tau^{k(\alpha-\beta)}}{\sigma^{k(\alpha-\beta)}}(\sigma-\tau)^{\tilde{\rho}-1}\,\mathrm{d}\sigma\\
&= c^2\left(\frac{\tau}{t}\right)^{k(\alpha-\beta)}(t-\tau)^{2\tilde{\rho}-1}B(\tilde{\rho},\tilde{\rho}).
\end{aligned}$$

Here we denoted by B the Euler Beta-function. It is then easily seen by induction that

$$\|h^{*n}(t,\tau)\|_{\beta\to 0} \le c^n\left(\frac{\tau}{t}\right)^{k(\alpha-\beta)}(t-\tau)^{n\tilde{\rho}-1}\frac{\Gamma(\tilde{\rho})^n}{\Gamma(n\tilde{\rho})}.$$

(δ) We may now use the last two steps to estimate the series for w. This gives

$$\begin{aligned}
\|w(t,\tau)\|_{\beta\to 0} &\le c\Gamma(\tilde{\rho})\left(\frac{\tau}{t}\right)^{k(\alpha-\beta)}(t-\tau)^{\tilde{\rho}-1}\sum_{n=1}^{\infty}\frac{(c\Gamma(\tilde{\rho})(t-\tau)^{\tilde{\rho}})^{n-1}}{\Gamma(n\tilde{\rho})}\\
&\le c\left(\frac{\tau}{t}\right)^{k(\alpha-\beta)}(t-\tau)^{\tilde{\rho}-1}
\end{aligned}$$

The convergence of the series follows from

$$\Gamma(n\tilde{\rho}) \ge \int_0^n \sigma^{n\tilde{\rho}-1}\,\mathrm{e}^{-\sigma}\,\mathrm{d}\sigma \ge \frac{1}{\tilde{\rho}}n^{n\tilde{\rho}-1}\,\mathrm{e}^{-n} \tag{1.15}$$

by taking the $(n-1)$th root of the last term of the above inequalities and observing that the result is a positive, strictly growing function of n.

(ε) By the first and the preceding steps we infer now that

$$\|U_A(t,\tau)\|_{\beta\to\alpha} \le \|a(t,\tau)\|_{\beta\to\alpha} + \int_\tau^t \|a(t,\sigma)\|_{0\to\alpha}\|w(\sigma,\tau)\|_{\beta\to 0}\,\mathrm{d}\sigma$$
$$\le c\frac{\tau^{k(\alpha-\beta)}}{(t-\tau)^{\alpha-\beta}} + c\int_\tau^t \frac{\sigma^{k\alpha}}{(t-\sigma)^\alpha}\frac{\tau^{k(\alpha-\beta)}}{\sigma^{k(\alpha-\beta)}}(\sigma-\tau)^{\tilde\rho-1}\,\mathrm{d}\sigma$$
$$\le c\frac{\tau^{k(\alpha-\beta)}}{(t-\tau)^{\alpha-\beta}},$$

provided $\beta < \tilde\rho$. This concludes the proof. ■

Remark 1.7. It will turn out that the condition $\beta < \tilde\rho$ is not restrictive as far as the solvability of singular Cauchy problems is concerned. We shall in fact see that β may always be chosen arbitrary small.

Remark 1.8. In the previous lemma the constant c only depends on J if the singular family is kept fixed. Otherwise it also depends on the Hölder norm of A (see estimate (1.12)). Nevertheless, with the help of Lemma 1.5, it is easily seen that it can be chosen to be independent of the particular singular family in a C_k^ρ-neighbourhood of A.

PROPOSITION 1.9. Let A be a singular family of generators. Then, for $t > 0$,

$$U_A(t,\tau) \to 0 \text{ in } \mathcal{L}(E_0, E_\alpha), \qquad \text{as } \tau \to 0,$$

for $\alpha \in [0, 1]$.

Proof. The stated convergence is an easy consequence of Lemma 1.6 provided $\alpha \in (0, 1]$. If $\alpha = 0$ we observe that, for $0 < 2\varepsilon \le t$,

$$\|U_A(t,\varepsilon)\|_{0\to 0} \le \|U_A(t,2\varepsilon)\|_{0\to 0}\|U_A(2\varepsilon,\varepsilon)\|_{0\to 0}.$$

Thus by Lemma 1.6 the assertion is proved, if we verify that

$$U_A(2\varepsilon,\varepsilon) \to 0 \text{ in } \mathcal{L}(E_0, E_0), \qquad \text{as } \varepsilon \to 0.$$

To see this we first infer from (1.11) that

$$\|a(2\varepsilon,\varepsilon)\|_{0\to 0} \le M\mathrm{e}^{-\omega_0/\varepsilon^{k-1}}.$$

On the assumption that

$$\|w(t,\tau)\|_{0\to 0} \le c(t-\tau)^{\tilde\rho-1}\exp\left(-\frac{\omega_0(t-\tau)}{4\tau^k}\right), \tag{1.16}$$

and following the steps of the proof of Lemma 1.6 we then obtain the following estimate for $a * w$:

$$\|a * w(2\varepsilon, \varepsilon)\|_{0\to 0} \leq \int_{\varepsilon}^{2\varepsilon} \|a(2\varepsilon, \sigma)\|_{0\to 0}\|w(\sigma, \varepsilon)\|_{0\to 0}\, d\sigma$$

$$\leq c\int_{\varepsilon}^{2\varepsilon} (2\varepsilon - \sigma)^{\tilde{\rho}-1}(\sigma - \varepsilon)^{\tilde{\rho}-1} \exp\left(-\frac{\omega_0(2\varepsilon - \sigma)}{4\varepsilon^k}\right) \exp\left(-\frac{\omega_0(\sigma - \varepsilon)}{4\varepsilon^k}\right) d\sigma$$

$$= c\varepsilon^{2\tilde{\rho}} \exp\left(-\frac{w_0}{4\varepsilon^{k-1}}\right).$$

Thus, since $U_A = a + a * w$, we are finished if we prove (1.16). To this end observe first that, for $\varepsilon \leq \sigma \leq 2\varepsilon$,

$$\|h(\sigma, \varepsilon)\|_{0\to 0} \leq c(|\sigma - \varepsilon|^{\tilde{\rho}-1} + 1/\varepsilon) \exp\left(-\frac{\omega_0(\sigma - \varepsilon)}{\varepsilon^k}\right).$$

This implies, as in the proof of the previous lemma, by means of estimates (1.13) and (1.14) that

$$\|h(\sigma, \varepsilon)\|_{0\to 0} \leq c|\sigma - \varepsilon|^{\tilde{\rho}-1} \exp\left(-\frac{\omega_0(\sigma - \varepsilon)}{2\varepsilon^k}\right).$$

An induction argument then gives

$$\|h^{*n}(\sigma, \varepsilon)\|_{0\to 0} \leq c\,\frac{\Gamma(\tilde{\rho})^n}{\Gamma(n\tilde{\rho})}(\sigma - \varepsilon)^{n\tilde{\rho}-1} \exp\left(-\frac{\omega_0(\sigma - \varepsilon)}{2\varepsilon^k}\right),$$

and hence

$$\|w(\sigma, \varepsilon)\|_{0\to 0} \leq \sum_{n=1}^{\infty} \|h^{*n}(\sigma, \varepsilon)\|_{0\to 0} \leq c(\sigma - \varepsilon)^{\tilde{\rho}-1} \exp\left(-\frac{\omega_0(\sigma - \varepsilon)}{4\varepsilon^k}\right).$$

The convergence of the series follows as in the previous lemma by (1.15). The proof is now complete. ■

It now makes sense to put the following definition.

Definition. Let A be a singular family of operators satisfying (1.1). Then we call

$$U_A(t, \tau) := \begin{cases} U_\tau(t, \tau), & \text{if } 0 < \tau \leq t, \\ 0 & \text{if } \tau = 0 \text{ and } t \geq 0 \end{cases}$$

the singular evolution operator generated by the singular family A.

Remark 1.10. Assume that the singular generator A has the special form

$$A(t) = \frac{1}{s(t)} B,$$

for some generator $-B$ of an exponentially decaying C_0-semigroup T_B and some function $s \in C^p(J, (0, \infty))$ satisfying "$s(t) \sim t^k$" for small times. Then the singular evolution operator U_A generated by A is given explicitly by

$$U_A(t, \tau) := \begin{cases} T_B\left(\int_\tau^t s^{-1}(\sigma)\, d\sigma\right), & \text{if } 0 < \tau \le t, \\ 0 & \text{if } \tau = 0 \text{ and } t \ge 0. \end{cases}$$

This follows immediately from the definition.

2. THE SINGULAR VARIATION-OF-CONSTANTS-FORMULA

Definition. We call a function $u: J \to E_0$ solution of the singular Cauchy problem $(1.3)_{(A,f)}$ if

(i) $$u \in C(J, E_0) \cap C^1(\dot{J}, E_0)$$

(ii) $$u(t) \in \operatorname{dom} A(t),\ t \in \dot{J} \text{ and } \dot{u}(t) + A(t)u(t) = f(t),\ t \in \dot{J}.$$

The collected properties of the singular evolution operator are enough to prove the following fundamental characterization theorem for the solutions of the singular Cauchy problem $(1.3)_{(A,f)}$.

Main proposition. Let A be a singular family of generators satisfying (1.1) and $f \in L_{1,\text{loc}}(J, E_0)$. Assume that $u: J \to E_0$ is a solution of the singular Cauchy problem $(1.3)_{(A,f)}$

$$\dot{u} + A(t)u = f(t), \qquad t \in \dot{J},$$

in the Banach space E_0. Then

$$u(t) = \int_0^t U_A(t, \tau) f(\tau)\, d\tau. \tag{2.1}$$

In other words, if a solution of $(1.3)_{(A,f)}$ exists then it is unique and is necessarily given by the singular variation-of-constants-formula.

Proof. It is well known that in the regular case the solution v of the following Cauchy problem $(\mathrm{CP})_{\delta,x}$

$$\begin{cases} \dot{u} + B(t)u = f(t), & t > \delta > 0, \\ u(\delta) = x \end{cases}$$

for a regular family $\{B(t) \mid t \in J\}$ is given by the variation-of-constants-formula, that is to say, by

$$v(t) = U_B(t, \delta)x + \int_\delta^t U_B(t, \tau) f(\tau)\, d\tau.$$

Thus, since, for $T \in J$, $u_\delta := u|_{[\delta, T]}$ solves $(\mathrm{CP})_{\delta, u(\delta)}$ for A we see that

$$u_\delta(t) = U_A(t, \delta)u(\delta) + \int_\delta^t U_A(t, \tau)f(\tau)\,\mathrm{d}\tau.$$

Let now $\delta \geq \delta' > 0$. Then, for $t \geq \delta$,

$$\begin{aligned} u_{\delta'}(t) &= U_A(t, \delta)U_A(\delta, \delta')u(\delta') + \int_{\delta'}^{\delta} U_A(t, \tau)f(\tau)\,\mathrm{d}\tau + \int_\delta^t U_A(t, \tau)f(\tau)\,\mathrm{d}\tau \\ &= U_A(t, \delta)\left(U_A(\delta, \delta')u(\delta') + \int_{\delta'}^{\delta} U_A(\delta, \tau)f(\tau)\,\mathrm{d}\tau\right) + \int_\delta^t U_A(t, \tau)f(\tau)\,\mathrm{d}\tau. \qquad (2.2) \end{aligned}$$

Thus, if we show that

$$u(\delta) = U_A(\delta, \delta')u(\delta') + \int_{\delta'}^{\delta} U_A(\delta, \tau)f(\tau)\,\mathrm{d}\tau, \tag{2.3}$$

we obtain

$$u_\delta(t) = u_{\delta'}(t), \qquad t \geq \delta, \delta \geq \delta' > 0.$$

To see this observe that

$$U_A(\delta, \delta')u(\delta') + \int_{\delta'}^{\delta} U_A(t, \tau)f(\tau)\,\mathrm{d}\tau$$

is the value at time δ of the solution of $(\mathrm{CP})_{\delta', u(\delta')}$. Since $u|_{[\delta', T]}$ is also a solution of the same Cauchy problem we conclude that (2.3) is valid. By Lemma 1.9 it follows now that

$$U_A(t, \delta)u(\delta) \to 0 \quad \text{in } E_0, \qquad \text{as } \delta \to 0,$$

since $\{u(\delta) \mid \delta \in \dot{J}\}$ is bounded in the same space by assumption. On the other hand, it follows from

$$U_A(t, \cdot) \in L_\infty(J, \mathcal{L}(E_1, E_0))$$

that $\int_0^t U_A(t, \tau)f(\tau)\,\mathrm{d}\tau$ exists and hence we see that

$$\int_\delta^t U_A(t, \tau)f(\tau)\,\mathrm{d}\tau \to \int_0^t U_A(t, \tau)f(\tau)\,\mathrm{d}\tau, \qquad \text{as } \delta \to 0.$$

Thus the assertion is implied by the following diagram:

$$\begin{array}{ccccc} u_\delta(t) = & U_A(t, \delta)u(\delta) & + & \displaystyle\int_\delta^t U_A(t, \tau)f(\tau)\,\mathrm{d}\tau \\ \| & \downarrow & \delta \to 0 & \downarrow \\ u(t) = & 0 & + & \displaystyle\int_0^t U_A(t, \tau)f(\tau)\,\mathrm{d}\tau. \end{array} \quad \blacksquare$$

Remark 2.1. It is an obvious consequence of the previous proposition that a solution of $(1.3)_{(A,f)}$ necessarily takes 0 as initial value.

The last proposition, as in regular theory, motivates the following definition of "mild solution".

Definition. We call $u: J \to E$ a mild solution of $(1.3)_{(A,f)}$ if

$$u(t) = \int_0^t U_A(t, \tau) f(\tau)\, \mathrm{d}\tau,$$

i.e. if the singular variation-of-constants-formula exists in E.

Proposition 2 also has a series of important consequences, which we partly summarize in the following.

COROLLARY 2.2. The singular variation-of-constants-formula implies that:

(i) The solution of the singular Cauchy problem $(1.3)_{(A,f)}$ enjoys all the nice regularizing properties of regular Cauchy problems of parabolic type.

(ii) As far as global existence is concerned, all the known general results for the parabolic case apply.

Proof. This is a consequence of the fact that, once the local existence for the singular Cauchy problem is established, the further analysis, i.e. for positive times, does not differ any longer from that of the regular parabolic case. More precisely it suffices then to consider the local solution u of the singular Cauchy problem $(1.3)_{(A,f)}$ as a solution of the parabolic Cauchy problem

$$\dot{v} + A(t)v = f(t), \quad t > \delta, \qquad v(\delta) = u(\delta),$$

where $\delta > 0$ is taken in the existence interval of u. This means that the variation-of-constants-formula established in Lemma 2 has "except at the origin" the same properties as parabolic propagators, that is to say, evolution operators for parabolic Cauchy problems. ■

Remark 2.3. For a comprehensive treatment of the topics we referred to in the preceding corollary we refer to the books [3, 7].

Next we prove an important smoothing property of the singular variation-of-constants-formula, on which the further analysis of the solvability of singular Cauchy problems will be based. We begin with a useful lemma.

LEMMA 2.4. Assume that A is a singular family of generators satisfying (1.1) and that $\alpha \in (0, 1)$. Then the following estimate

$$\|U_A(t, \tau) - U_A(r, \tau)\|_{0 \to \alpha} \leq c(t, r, \tau)\tau^{k\alpha}(t - r)^{\varepsilon}$$

is valid for the corresponding singular propagator U_A if $\varepsilon < \alpha \wedge (1 - \alpha)$, where

$$c(t, r, \tau) = \mathrm{const}\left[\frac{(t - r)^{\alpha - \varepsilon}}{(t - \tau)^{\alpha}(r - \tau)^{\alpha}} + (t - r)^{1 - \alpha - \varepsilon}(r - \tau)^{1 - \tilde{\rho}} + 1\right].$$

Proof. Assume without loss of generality that $r \le t$. Then

$$\begin{aligned}\|U_A(t,\tau) - U_A(r,\tau)\|_{0\to\alpha} &\le \|a(t,\tau) - a(r,\tau)\|_{0\to\alpha} \\ &\quad + \|a * w(t,\tau) - a * w(r,\tau)\|_{0\to\alpha} \\ &=: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Let us first estimate I.

$$\begin{aligned}\mathrm{I} &= \|\exp(-(t-\tau)A(\tau)) - \exp(-(r-\tau)A(\tau))\|_{0\to\alpha} \\ &= \left\| T_{\tau^k A(\tau)}\left(\frac{t-\tau}{\tau^k}\right) - T_{\tau^k A(\tau)}\left(\frac{r-\tau}{\tau^k}\right)\right\|_{0\to\alpha} \\ &\le \int_{(r-\tau)/\tau^k}^{(t-\tau)/\tau^k} \|\tau^k A(\tau) T_{\tau^k A(\tau)}(\sigma)\|_{0\to\alpha}\,\mathrm{d}\sigma \\ &\le c\,\frac{\tau^{k\alpha}(t-r)^\alpha}{(t-\tau)^\alpha (r-\tau)^\alpha}.\end{aligned}$$

Hereby we wrote T_B to indicate the semigroup generated by the operator $-B$. Furthermore we made use of the semigroup estimate

$$\|BT_B(t)\|_{0\to\alpha} \le ct^{-1-\alpha},$$

which is obtained by interpolation from

$$\|BT_B(t)\|_{0\to 0} \le ct^{-1},$$

and

$$\|BT_B(t)\|_{0\to 1} = \|B^2 T_B(t)\|_{0\to 0} \le ct^{-2}.$$

As to the second term we have

$$\begin{aligned}\mathrm{II} &= \left\| \int_\tau^t a(t,\sigma) w(\sigma,\tau)\,\mathrm{d}\sigma - \int_\tau^r a(r,\sigma) w(\sigma,\tau)\,\mathrm{d}\sigma \right\|_{0\to\alpha} \\ &\le \left\| \int_\tau^t a(t,\sigma) w(\sigma,\tau)\,\mathrm{d}\sigma \right\|_{0\to\alpha} + \left\| \int_\tau^r \{a(t,\sigma) - a(r,\sigma)\} w(\sigma,\tau)\,\mathrm{d}\sigma \right\|_{0\to\alpha} \\ &=: \mathrm{III} + \mathrm{IV}.\end{aligned}$$

Then by (1.11) and step (δ) in the proof of Lemma 1.6

$$\mathrm{III} \le c \int_r^t \frac{\sigma^{k\alpha}}{(t-\sigma)^\alpha} \frac{\tau^{k\alpha}}{\sigma^{k\alpha}} (\sigma-\tau)^{\tilde\rho - 1}\,\mathrm{d}\sigma = c\,\frac{\tau^{k\alpha}(t-r)^{1-\alpha}}{(r-\tau)^{1-\tilde\rho}}.$$

Observing now that

$$\|a(t,\sigma) - a(r,\sigma)\|_{0\to\alpha} \le c\,\frac{\sigma^{k\alpha}(t-r)^\alpha}{(t-\sigma)^\alpha (r-\sigma)^\alpha}$$

is implied by the established estimate for I, we arrive at

$$\begin{aligned}
\mathrm{IV} &\leq \int_\tau^r \|a(t,\sigma) - a(r,\sigma)\|_{0\to\alpha} \|w(\sigma,\tau)\|_{0\to 0}\,\mathrm{d}\sigma \\
&\leq c(t-r)^\alpha \tau^{k\alpha} \int_\tau^r \frac{\sigma^{k\alpha}}{(t-\sigma)^{\alpha-\varepsilon+\varepsilon}(r-\sigma)^\alpha} \frac{(\sigma-\tau)^{\tilde{\rho}-1}}{\sigma^{k\alpha}}\,\mathrm{d}\sigma \\
&\leq c(t-r)^\varepsilon \tau^{k\alpha} \int_\tau^r \frac{(\sigma-\tau)^{\tilde{\rho}-1}}{(r-\sigma)^{\alpha+\varepsilon}}\,\mathrm{d}\sigma \\
&= c\tau^{k\alpha}(t-r)^\varepsilon B(1-\alpha-\varepsilon, \tilde{\rho}),
\end{aligned}$$

which is valid for each $\varepsilon < 1 - \alpha$ and $\tilde{\rho} < \rho \wedge 1/k'$. From this we infer that

$$\mathrm{II} \leq c\tau^{k\alpha}\{(r-\tau)^{\tilde{\rho}-1}(t-r)^{1-\alpha} + (t-r)^\varepsilon\}.$$

Thus, for $\varepsilon < \alpha \wedge (1-\alpha)$, we obtain that

$$\|U_A(t,\tau) - U_A(r,\tau)\|_{0\to\alpha} \leq c(t,r,\tau)\tau^{k\alpha}(t-r)^\varepsilon,$$

and the proof is complete. ■

Now we can prove the following.

Proposition 2.5. Assume that $f \in C_{k-1}(J, E_0)$ and that A is a singular family of generators satisfying (1.1). Then

$$u(\cdot) = \int_0^{\cdot} U_A(\cdot,\tau) f(\tau)\,\mathrm{d}\tau \in C^\nu(J, E_\alpha),$$

for $\alpha \in [1/k', 1)$ and $0 \leq \nu < \alpha \wedge (1-\alpha)$, where k' is the dual exponent of k.

Proof. We divide the proof into two steps.

(α) Let us first consider continuity at zero. From

$$\|u(t)\|_\alpha = \left\| \int_0^t U_A(t,\tau) f(\tau)\,\mathrm{d}\tau \right\|_\alpha \leq \int_0^t \|U_A(t,\tau)\|_{0\to\alpha} \|f(\tau)\|_0\,\mathrm{d}\tau$$

we obtain by Lemma 1.6 and by

$$\|f(\tau)\|_0 \leq \frac{1}{\tau^{k-1}} \|f(\tau)\|_{C_{k-1}}, \tag{2.4}$$

that

$$\|u(t)\|_\alpha \leq c \int_0^t \frac{\tau^{k\alpha-k+1}}{(t-\tau)^\alpha}\,\mathrm{d}\tau = t^{(k-1)(\alpha-1)+1} B(1-\alpha, ka-k+2), \tag{2.5}$$

which is the desired Hölder continuity at zero, since $(k-1)(\alpha-1) + 1 \geq 1 - \alpha > \varepsilon$ provided $\alpha \geq 1/k'$.

(β) Let now $0 < r \le t \in J$. Consider

$$\begin{aligned}\|u(t) - u(r)\|_\alpha &\le \int_r^t \|U_A(t, \tau)\|_{0\to\alpha}\|f(\tau)\|_0 \,\mathrm{d}\tau \\ &\quad + \int_0^r \|U_A(t, \tau) - U_A(r, \tau)\|_{0\to\alpha}\|f(\tau)\|_0 \,\mathrm{d}\tau \\ &:= \mathrm{I} + \mathrm{II}.\end{aligned}$$

Then Lemma 1.6 and (2.4) imply, provided $\alpha \ge 1/k'$ (which is no restriction), that

$$\mathrm{I} \le c\int_r^t \frac{\tau^{k\alpha}}{\tau^{k-1}(t-\tau)^\alpha}\|f\|_{C_{k-1}}\,\mathrm{d}\tau = c\|f\|_{C_{k-1}} t^{k\alpha-k+1}(t-r)^{1-\alpha}, \tag{2.6}$$

and, on the other hand, we see by Lemma 2.4 that

$$\begin{aligned}\mathrm{II} &\le c\int_0^r c(r,t,\tau)(t-r)^\varepsilon \tau^{k\alpha-k+1}\|f(\tau)\|_{C_{k-1}}\,\mathrm{d}\tau \\ &= c(t-r)^\varepsilon \int_0^r \left\{\frac{(t-r)^{\alpha-\varepsilon}}{(t-\tau)^{\alpha-\varepsilon+\varepsilon}(r-\tau)^\alpha} + \frac{(t-r)^{1-\alpha-\varepsilon}}{(r-\tau)^{1-\tilde{\rho}}} + c\right\}\tau^{k\alpha-k+1}\,\mathrm{d}\tau \\ &\le c(t-r)^\varepsilon\{r^{(k-1)\alpha+2-k-\varepsilon} + r^{k\alpha+\tilde{\rho}-k+1} + r^{k(\alpha-1)+2}\},\end{aligned} \tag{2.7}$$

for each $\varepsilon < \alpha \wedge (1-\alpha)$, since it is easily verified that

$$(k-1)\alpha + 2 - k - \varepsilon > 0, \quad k\alpha + \tilde{\rho} - k + 1 > 0, \quad k(\alpha - 1) + 2 > 0,$$

provided $\alpha \ge 1/k'$. Hence the assertion is proved. ■

Remark 2.6. It is easily seen, looking carefully at the estimates in the above proof, that the assumption on f may be relaxed to

$$f \in C_l(J, E_0) \text{ for some } 0 \le l < k.$$

In this case one would merely have to adapt the parameters $\alpha, \nu \in (0, 1)$.

Remark 2.7. It has to be pointed out that singular evolution operators have a better regularizing property than regular ones. More precisely we have Hölder continuity into each intermediate space up to the initial time, whereas in the regular case this can only be obtained with a loss of spacial regularity (cf. [8, in particular Theorem 5.3]). This will turn out to be useful in the existence-proof for quasilinear singular Cauchy problems.

COROLLARY 2.8. Assume that $f \in C_{k-1}(J, E_0)$ and that A is a singular family of generators satisfying (1.1). Then there exist a neighbourhood $\mathfrak{U}$ of (A, f) in the natural product topology and $\varrho, \delta \in (0, 1)$ with

$$\|u(t) - u(r)\|_\alpha \le ct^\delta|t - r|^\varrho, \qquad 0 \le r \le t \in J,$$

for the solution u of $(1.2)_{(B,g)}$ for arbitrary $(B, g) \in \mathfrak{U}$.

Proof. Fix $\alpha \geq 1/k'$. Then we infer from (2.5), (2.6) and (2.7) that

$$\|u(t) - u(r)\|_\alpha \leq ct^\delta |t - r|^\varrho,$$

if we choose $0 < \varrho < (1 - \alpha) \wedge \alpha$ and

$$\delta = \{(k-1)\alpha + 2 - k - \varepsilon\} \wedge \{k\alpha + \tilde{\rho} - k + 1\} \wedge \{k(\alpha - 1) + 2\} \wedge \{(k-1)(\alpha - 1) + 1 - \varrho\},$$

where $\tilde{\rho} < \rho \wedge 1/k'$. It is easily verified that δ has to be positive. The existence of the neighborhood $\mathfrak{U}$, in which the estimate can be made independent of (B, g), follows easily for g and from Remark 1.8 for B. ■

3. QUASILINEAR SINGULAR CAUCHY PROBLEMS

In this section we devote our interest to the discussion of the quasilinear version of $(1.3)_{(A,f)}$. A local existence result will be proved. Consider the following

$$\dot{u} + A(t, u)u = F(t, u), \qquad t \in \dot{J}, \tag{3.1$_{(A,F)}$}$$

in the Banach space E_0. Given $\alpha \in (0, 1)$ and $0 \in X_\alpha \overset{\circ}{\subset} E_\alpha$ we assume that

$$A \in C_k^{\varrho, 1-}(\times X_\alpha, \mathcal{H}^-(E_1, E_0)) \tag{3.2}$$

and that

$$F \in C_{k-1,b}^{0,1-}(J \times X_\alpha, E_\beta) \text{ for } 0 < \beta < \alpha < 1. \tag{3.3}$$

A few remarks about the notation are in order. By $C_k^{\varrho,1-}$ we mean C_k^ϱ in the first variable and C^{1-} in the second. The notation $C_{k-1}^{0,1-}$ has to be interpreted in an analogous manner. The additional subscript b corresponds to the requirement that F be bounded on bounded subsets. Lastly the symbol $\overset{\circ}{\subset}$ means "open subset".

Definition. A function $u: J \to X_\alpha$ is said to be a solution of $(3.1)_{(A,F)}$ on J if

(i) $$u \in C(J, E_0) \cap C^1(\dot{J}, E_0),$$

(ii) $u(t) \in \operatorname{dom} A(t, u(t)),\ t \in \dot{J}$ and $\dot{u}(t) + A(t, u(t))u(t) = F(t, u(t)),\ t \in \dot{J}$.

LEMMA 3.1. Let A and B be singular families of generators satisfying (1.1). Then the following estimate

$$\|U_A(t, \tau) - U_B(t, \tau)\|_{\beta \to \alpha} \leq c \frac{\tau^{k(\alpha - \beta)}}{(t - \tau)^{(\alpha - \beta)}} \|A - B\|_{C_k}, \qquad 0 \leq \tau \leq t \in J.$$

is valid for the corresponding evolution operators. The constant c may be chosen to be independent of B in a neighbourhood of A.

Proof. It follows directly from the determining integral equation (1.4) that

$$U_A(t, \tau) - U_B(t, \tau) = \int_\tau^t U_A(t, \sigma)[A(\sigma) - B(\sigma)]U_B(\sigma, \tau)\, d\sigma.$$

This entails by Lemma 1.6 that

$$\begin{aligned}\|U_A(t,\tau) - U_B(t,\tau)\|_{\beta\to\alpha} &\le \int_\tau^t \|U_A(t,\sigma)\|_{0\to\alpha}\|A(\sigma) - B(\sigma)\|_{1\to 0}\|U_B(\sigma,\tau)\|_{\beta\to 1} \\ &\le c\|A - B\|_{C_k} \int_\tau^t \frac{\sigma^{k\alpha}}{(t-\sigma)^\alpha}\frac{1}{\sigma^k}\frac{\tau^{k(1-\beta)}}{(\sigma-\tau)^{1-\beta}}\,\mathrm{d}\sigma \\ &\le c\frac{\tau^{k(\alpha-\beta)}}{(t-\tau)^{\alpha-\beta}}\|A - B\|_{C_k}.\end{aligned}$$

The additional assertion is implied by Lemma 1.5. ■

Remark 3.2. It is worth while noticing that the assumption $\beta > 0$ is only needed to consider the quasilinear case, as the previous lemma shows. The proof of the next lemma will namely show that $\beta = 0$ is admissible in the semilinear case, where one does not need continuous dependence on A to prove existence.

LEMMA 3.3. Assume that A and B are singular families of generators satisfying condition (1.1) and that $f, g \in C_{k-1}(J, E_\beta)$. Denote by k' the dual exponent of k. Let $\alpha \in [1/k', 1)$ and $\beta \in (0, \rho \wedge 1/k')$ be such that $\alpha - \beta > (k-2)/(k-1)$, if $k > 2$. Let, moreover, u and v be the solutions of $(1.3)_{(A,f)}$ and $(1.2)_{(B,g)}$, respectively. Then

$$\|u(t) - v(t)\|_\alpha \le ct^{(k-1)(\alpha-\beta)-k+2}(\|A - B\|_{C_k} + \|f - g\|_{C_{k-1}}), \qquad 0 \le t \in J.$$

Proof. Observe first that

$$u(t) = \int_0^t U_A(t,\tau)f(\tau)\,\mathrm{d}\tau,$$

$$v(t) = \int_0^t U_B(t,\tau)g(\tau)\,\mathrm{d}\tau$$

by Lemma 2. Then we infer from Lemmas 1.6 and 3.1 that

$$\begin{aligned}\|u(t) - v(t)\|_\alpha &\le \int_0^t \|U_A(t,\tau) - U_B(t,\tau)\|_{\beta\to\alpha}\|f(\tau)\|_\beta\,\mathrm{d}\tau \\ &\quad + \int_0^t \|U_B(t,\tau)\|_{\beta\to\alpha}\|f(\tau) - g(\tau)\|_\beta\,\mathrm{d}\tau \\ &\le c\{\|A - B\|_{C_k}\|f\|_{C_{k-1}} + \|f - g\|_{C_{k-1}}\}\int_0^t \frac{\tau^{k(\alpha-\beta)-k+1}}{(t-\tau)^{\alpha-\beta}}\,\mathrm{d}\tau \\ &= ct^{(k-1)(\alpha-\beta)-k+2}(\|A - B\|_{C_2}\|f\|_{C_1} + \|f - g\|_{C_1}).\end{aligned}$$

Thus the proof is complete. ■

Remark 3.4. The constant c in the claim of the last lemma depends obviously on A, B, f and g. It is, however, locally uniformly bounded by Lemma 1.5.

For ϱ chosen such as to satisfy the condition of Corollary 2.8 and for functions $u \in C^{\varrho}(J, E_\alpha)$ we now define $\Phi(u)$ by

$$\Phi(u)(t) := \int_0^t U_{A(\cdot,u)}(t, \tau)F(\tau, u(\tau))\,\mathrm{d}\tau, \qquad t \in J.$$

In the following lemma we prove two important properties of Φ.

LEMMA 3.5. Suppose that A satisfies assumption (3.2), F assumption (3.3) and let $\delta \in (0, 1)$. Choose α and β as in Lemma 3.3. Then there exists $R > 0$ such that, for $u, v \in \bar{\mathbb{B}}_{C^\delta(J,E_\alpha)}(0, R)$, the estimates

$$\|\Phi(u)(t)\|_\alpha \leq ct^{(k-1)(\alpha-\beta)-k+2}$$

and

$$\|\Phi(u(t) - \Phi(v)(t)\|_\alpha \leq ct^{(k-1)(\alpha-\beta)-k+2}\|u - v\|_{C(J,E_\alpha)}$$

are valid for a positive constant c independent of $u, v \in \bar{\mathbb{B}}_{C^\delta(J,E_\alpha)}(0, R)$.

Proof. By Lemma 1.5 and Remark 1.8 there exists a neighbourhood of the set

$$[t \mapsto A(t, 0)]$$

in $C_k^{\varrho\wedge\delta}$ such that uniform estimates hold for the evolution operators corresponding to generators in this neighbourhood. It follows from hypothesis (3.2) that we find a real number $R > 0$ such that uniform estimates are valid for $U_{A(\cdot,u)}$, if $u \in \bar{\mathbb{B}}_{C^\delta(J,E_\alpha)}(0, R)$. By assumption (3.3) we may assume without loss of generality that this also holds for $F(\cdot, u(\cdot))$ with the same R. Thus for $u, v \in \bar{\mathbb{B}}_{C^\delta(J,E_\alpha)}(0, R)$, we see, on the one hand, that

$$\begin{aligned}\|\Phi(u)(t)\|_\alpha &\leq \int_0^t \|U_{A(\cdot,u)}(t, \tau)\|_{\beta\to\alpha}\|F(\tau, u(\tau))\|_\beta\,\mathrm{d}\tau \\ &\leq c\int_0^t \frac{\tau^{k(\alpha-\beta)-k+1}}{(t-\tau)^{\alpha-\beta}}\,\mathrm{d}\tau = ct^{(k-1)(\alpha-\beta)-k+2}.\end{aligned}$$

On the other hand we have, by Lemma 3.3 and the hypotheses, that the following estimate is valid:

$$\begin{aligned}&\|\Phi(u)(t) - \Phi(v)(t)\|_\alpha \\ &\quad\leq ct^{(k-1)(\alpha-\beta)-k+2}\{\|A(\cdot, u) - A(\cdot, v)\|_{C_k} + \|F(\cdot, u) - F(\cdot, v)\|_{C_{k-1}}\} \\ &\quad\leq ct^{(k-1)(\alpha-\beta)-k+2}\|u - v\|_{C(J,E_\alpha)}.\end{aligned}$$

The claim is thus proved. ∎

We can now state and prove the following main theorem.

MAIN THEOREM. Assume that A and F satisfy conditions (3.2) and (3.3), respectively. Then there exists $T > 0$ such that the quasilinear singular Cauchy problem in E_0

$$\dot{u} + A(t, u)u = F(t, u), \qquad t \in \dot{J} \tag{3.4}$$

has a unique local solution $u \in C([0, T], X_\alpha)$. It is the unique fixed point of the singular variation-of-constants-formula

$$u(t) = \int_0^t U_{A(\cdot, u)}(t, \tau)F(\tau, u(\tau))\, d\tau.$$

Moreover the solution u enjoys the following regularity properties

$$C^\delta([0, T], E_\varepsilon) \cap C((0, T], E_1) \cap C^1((0, T], E_0),$$

provided $\varepsilon, \delta \in (0, 1)$ with $\varepsilon \geq 1/k'$ and $\delta < \varepsilon \wedge (1 - \varepsilon)$.

Proof. Lemma 3.5 and Corollary 2.8 imply the existence of constants $R > 0$ and $\varrho > 0$, respectively, with the properties stated there. Define

$$X_T := \{u \in \bar{\mathbb{B}}_{C^\varrho([0,T],E_\alpha)}(0, R) \mid u(0) = 0\}.$$

Then we infer again from Lemma 3.5 and Corollary 2.8 that, provided T is chosen small enough, Φ becomes a self-map on X_T. Lemma 3.5 entails furthermore that Φ is contractive on X_T with respect to the topology induced by $C(J, E_\alpha)$. To attain this we may possibly once more make T smaller. Since $\bar{\mathbb{B}}_{C^\varrho([0,T],E_\alpha)}(0, R)$ is complete with respect to the topology of $C(J, E_\alpha)$, we argue, by means of Banach's fixed point theorem, that there exists a unique fixed point $\bar{u}$ for Φ. We are thus left with proving that $\bar{u}$ is actually a solution of $(3.1)_{(A,F)}$ and belongs to the claimed regularity classes. Notice that, in fact, the proof of the main proposition implies that

$$v(t) := \int_0^t U_{A(\cdot, \bar{u})}(t, \tau)F(\tau, \bar{u}(\tau))\, d\tau$$

coincides on $[\varepsilon, T]$, for $0 < \varepsilon < T$, with the solution u_ε of

$$\dot{u} + A(t, \bar{u})u = F(t, \bar{u}), \qquad t \in (\varepsilon, T]$$

$$u(\varepsilon) = \bar{u}(\varepsilon)$$

so that $v = \Phi(\bar{u}) = \bar{u}$ is indeed a solution of the above regular problem in $(\varepsilon, T]$ and enjoys by Corollary 2.2 (cf. also Theorems II.1.2.1 and II.1.2.2 in [3]) the desired regularity properties, i.e.

$$\bar{u} \in C((\varepsilon, T], E_1) \cap C^1((\varepsilon, T], E_0).$$

The arbitrariness of the choice of ε implies now that $\bar{u}$ is a solution of $(3.1)_{(A,F)}$ in $(0, T]$ and enjoys the asserted regularities. The claimed Hölder continuity follows from Proposition 2.5. ∎

4. SINGULAR INITIAL BOUNDARY VALUE PROBLEMS

In this concluding section we consider a first application of the results concerning abstract singular Cauchy problems of the previous sections. Of concern are singular initial boundary value problems of parabolic type. First we need to introduce some terminology and a fundamental generation theorem for second order ellipticc boundary value problems. The aim being a major ease in the formulation and analysis of singular initial boundary value problems thereafter. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary $\partial\Omega$. Assume $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are disjoint and open in $\partial\Omega$. Note that the cases $\Gamma_0 = \varnothing$ and $\Gamma_1 = \varnothing$ are of course not excluded, whereas the case $\partial\Omega = \varnothing$ is. If we define

$$\delta(x) := \begin{cases} 1, & x \in \Gamma_1, \\ 0, & x \in \Gamma_0, \end{cases}$$

then $\delta \in C(\partial\Omega, \{0, 1\})$. Further we denote by ν the outer unit normal to the boundary. For the trace operator we use the notation γ_∂. Fix now an increasing function $\sigma\colon [0, 1] \to [0, 1]$ with $\sigma(0) = 0$ and $\sigma(t) > t$, $0 < t < 1$, put $\rho(\alpha) := \sigma(|2\alpha - 1|)$, $0 \le \alpha \le 1$, and define for the same values of α

$$\mathbb{E}^\alpha(\Omega) := C^{\rho(\alpha)}(\bar{\Omega}, \mathbb{K})^{n^2+n+n} \times L_\infty(\Omega, \mathbb{K}) \times C^{\rho(\alpha)}(\partial\Omega, \mathbb{K}).$$

We observe that $\mathbb{E}^\alpha(\Omega) = \mathbb{E}^{1-\alpha}(\Omega)$, $0 \le \alpha \le 1$, and

$$\mathbb{E}^1(\Omega) \hookrightarrow \mathbb{E}^\alpha(\Omega) \hookrightarrow \mathbb{E}^\beta(\Omega) \hookrightarrow \mathbb{E}^{1/2}(\Omega), \qquad 1/2 < \beta < \alpha < 1.$$

Given $(a_{jk}, a_j, b_j, a_0, c) \in \mathbb{E}^\alpha(\Omega)$ with $a_{jk} = a_{kj}$ $(i, k \in \{1, \ldots, n\})$ we define the differential operator $\mathfrak{A}$ through

$$\mathfrak{A}u := -\partial_j(a_{jk}\partial_k u + a_j u) + b_j \partial_j u + a_0 u$$

and the boundary operator $\mathfrak{B}$ through

$$\mathfrak{B}u := \delta\{\nu^j \gamma_\partial(a_{jk}\partial_k u + a_j u) + c\gamma_\partial u\} + (1 - \delta)\gamma_\partial u.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are given as above, then the pair $(\mathfrak{A}, \mathfrak{B})$ is called boundary value problem of order at most two. By $(\mathfrak{A}^\#, \mathfrak{B}^\#)$ we further denote the formal adjoint boundary value problem to $(\mathfrak{A}, \mathfrak{B})$. It is the boundary value problem determined by the coefficients $(a_{jk}, b_j, a_j, a_0, c)$.

Terminology. From now on we no longer distinguish between a boundary value problem $(\mathfrak{A}, \mathfrak{B})$ and its coefficients in $\mathbb{E}^\alpha(\Omega)$. We thus write, for instance, $(\mathfrak{A}, \mathfrak{B}) \in \mathbb{E}^\alpha(\Omega)$ to mean that the coefficients of the boundary value problem are in those regularity classes.

Notation. The subset of $\mathbb{E}^\alpha(\Omega)$ consisting of those boundary value problems $(\mathfrak{A}, \mathfrak{B})$ which are in addition uniformly strongly elliptic is denoted by $\mathcal{E}^\alpha(\Omega)$.

For later purposes we also need to introduce some spaces related to a given boundary value problem $(\mathfrak{A}, \mathfrak{B})$. Given $p \in (1, \infty)$ put first

$$W_p^s(\Omega) := \begin{cases} B_{p,p}^s(\Omega), & s \in \mathbb{R}\backslash\mathbb{Z}, \\ H_p^s(\Omega), & s \in \mathbb{Z}, \end{cases} \tag{4.1}$$

where $B^s_{p,p}(\Omega)$ and $H^s_p(\Omega)$ are the Besov spaces and the classical Sobolev spaces, respectively. A precise definition and many important properties of these spaces can be found, for instance, in the book of Triebel [9]. Then, given $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}(\Omega) := \mathcal{E}^0(\Omega)$, we set

$$W^s_{p,\mathcal{B}} := \begin{cases} \{u \in W^s_p(\Omega) \mid \mathcal{B}u = 0\}, & s \in (1 + 1/p, 2], \\ \{u \in W^s_p(\Omega) \mid (1 - \delta)\gamma_\partial u = 0\}, & s \in (1/p, 1 + 1/p), \\ W^s_p(\Omega), & s \in [0, 1/p), \end{cases} \tag{4.2}$$

if $\delta \notin \{0, 1\}$. If δ vanishes identically then we extend the above definition to $s = 1 + 1/p$ with $\{u \in W^s_p \mid \gamma_\partial u = 0\}$. If $\delta \equiv 1$ to $s = 1/p$ with $W^{1/p}_p$. Let p' be the dual exponent to p. Then we lastly define

$$W^s_{p,\mathcal{B}} := \begin{cases} W^{-s}_p(\Omega), & s \in (-1 + 1/p, 0], \\ \{v \in W^{-s}_{p'}(\Omega) \mid (1 - \delta)\gamma_\partial v = 0\}', & s \in (-2 + 1/p, -1 + 1/p), \\ \{v \in W^{-s}_{p'}(\Omega) \mid \mathcal{B}^\# v = 0\}', & s \in [-2, -2 + 1/p), \end{cases} \tag{4.3}$$

if $\delta \notin \{0, 1\}$. Of course it follows that $W^s_{p,\mathcal{B}} = (W^{-s}_{p',\mathcal{B}^\#})'$. Thus we complete the scale in the obvious way if $\delta \in \{0, 1\}$, whenever $W^{-s}_{p',\mathcal{B}^\#}$ is defined.

Remark 4.1. It follows from the above definitions that, for $s \in (0, 2)$,

$$W^s_{p,\mathcal{B}} \overset{d}{\hookrightarrow} L_p \overset{d}{\hookrightarrow} W^{-s}_{p,\mathcal{B}}.$$

Remark 4.2. It is of great practical relevance that the above introduced spaces $W^s_{p,\mathcal{B}}$ depend, for $s \in (-2 + 1/p, 1 + 1/p)$, exclusively on the boundary map δ. This will play a central role in the discussion of initial boundary value problems, where the boundary operator $\mathcal{B}$ might depend on the time variable.

Remark 4.3. It is to be noted that the above introduced concrete spaces may also be recovered by interpolation (except for finitely many exceptional values of "s") between $L_p(\Omega)$ and $W^2_{p,\mathcal{B}}(\Omega) \doteq D(A)$ for positive values of s, and between $W^{-2}_{p,\mathcal{B}}(\Omega)$ and $L_p(\Omega)$, for negative values. For the precise statements we refer to [10] (in particular Sections 6, 7).

Next we turn to the definition of the realizations of $(\mathcal{A}, \mathcal{B})$ in the spaces of the above scale. We first need to introduce the concept of Dirichlet form.

Definition. Let $(\mathcal{A}, \mathcal{B}) \in \mathbb{E}(\Omega)$ be given. Then we call

$$\mathbf{a}(u, v) := \langle \partial_j v, a_{jk}\partial_k u + a_j u\rangle + \langle v, b_j \partial_j u + a_0 u\rangle + \langle \gamma_\partial v, c\gamma_\partial u\rangle_\partial, \tag{4.4}$$

the Dirichlet form associated to $(\mathcal{A}, \mathcal{B})$. Here we denoted by

$$\langle \cdot, \cdot \rangle_\partial := \int_{\partial\Omega} \gamma_\partial v \gamma_\partial u \, d\sigma_\partial$$

the duality pairing on the boundary of Ω and by σ_∂ the boundary measure on $\partial\Omega$.

It is easily seen that

$$\mathbf{a} \in \mathcal{L}(W^{2\beta}_{p',\mathcal{B}^{\#}}, W^{2\alpha}_{p,\mathcal{B}}; \mathbb{K}).$$

Definition. Suppose that $p \in (1, \infty)$ and $(\mathcal{A}, \mathcal{B}) \in \mathbb{E}^{\alpha}(\Omega)$. Then we define

$$A_{\alpha-1} \in \mathcal{L}(W^{2\alpha}_{p,\mathcal{B}}, W^{2\alpha-2}_{p,\mathcal{B}}),$$

the $W^{2\alpha-2}_{p,\mathcal{B}}$-realization of $(\mathcal{A}, \mathcal{B})$, as follows:

$$A_{\alpha-1} := \begin{cases} \mathcal{A}|_{W^{2\alpha}_{p,\mathcal{B}}}, & \text{if } 2\alpha \in (\|\delta\|_{\infty} + 1/p, 2], \\ [u \mapsto \mathbf{a}(\cdot, u)], & \text{if } 2\alpha \in (1/p, 1 + 1/p). \end{cases}$$

In other words, given $2\alpha \in (1/p, 1 + 1/p)$, $A_{\alpha-1}$ is the unique linear and continuous operator induced by the Dirichlet form $\mathbf{a}$ associated to $(\mathcal{A}, \mathcal{B})$, that is:

$$\langle v, A_{\alpha-1} u\rangle = \mathbf{a}(v, u), \qquad (v, u) \in W^{2-2\alpha}_{p',\mathcal{B}^{\#}} \times W^{2\alpha}_{p,\mathcal{B}}.$$

Remark 4.4. It can be shown with the help of Green's formulas that the definition is unambiguous for $2\alpha \in (1/p, 1 + 1/p)$ (cf. [10]).

We can now state the following fundamental.

THEOREM 4.5. Suppose $p \in (1, \infty)$ and $2\alpha \in [1, 2]$. If $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^{\alpha}(\Omega)$ then

$$A_{\beta-1} \in \mathcal{H}(W^{2\beta}_{p,\mathcal{B}}, W^{2\beta-2}_{p,\mathcal{B}}), \tag{4.5}$$

and $A_{\beta-1}$ has a compact resolvent provided $2\beta \in [2 - 2\alpha, 2\alpha] \cap (1/p, 2]$.

The proof of this important generation result can be found in [11] and [12].

Remark 4.6. It is known that $W^2_p \hookrightarrow\hookrightarrow L_p$. This fact implies that $A_{\alpha-1}$ has a compact resolvent for $\alpha \in [0, 1]$.

We are now ready to concentrate our attention on singular initial boundary value problems. From Theorem 4.5 we know when a boundary value problem $(\mathcal{A}, \mathcal{B})$ induces a generator A of an analytic semigroup on L_p (for some $p \in (1, \infty)$). Since we work in the class of exponentially decaying semigroups we should be able to tell when the generated semigroup possesses this property. The next remark shows that easy-to-verify conditions can be found, which ensure the decay of the semigroup.

Remark 4.7. It can be verified that the semigroup generated by the $W^{2\beta-2}_{p,\mathcal{B}}$-realization $A_{\beta-1}$ of $(\mathcal{A}, \mathcal{B}) \in \mathcal{E}^{\varrho}(\Omega)$ is exponentially decaying if, for instance, $a_j = b_j = a_0 = 0$, $\delta \not\equiv 1$ or if $\delta \equiv 1$ and $c \not\equiv 0$.

Let now $J \subset \mathbb{R}^+$ be a perfect interval containing the origin. Let, moreover, $2\gamma \in (1/p, 2]$ and $k > 1$. Assume that

$$(\mathcal{A}, \mathcal{B}) \in C^{\varrho}_{k}(J, \mathcal{E}^{\varrho}(\Omega)) \tag{4.6}$$

for some $2\varrho \in [1, 2]$ with $2\gamma \in [2 - 2\varrho, \varrho]$ and that

$$A_{\gamma-1}(t) \in \mathcal{H}^{-}(W^{2\gamma}_{p,\mathcal{B}(t)}, W^{2\gamma-2}_{p,\mathcal{B}(t)}) \tag{4.7}$$

for $t \in \dot{J}$. Lastly let $\sigma \in [0, 1)$ and assume that

$$(f, g) \in C^{\sigma}_{k-1}(J, W^{2\gamma-2}_{p,\mathfrak{B}} \times \partial W^{2\gamma}_p). \tag{4.8}$$

Observe that the latter condition makes sense since the spaces $W^{2\gamma-2}_{p,\mathfrak{B}}$ do not depend on time in the given range of 2γ. Consider the following singular initial boundary value problem:

$$\dot{u} + \mathfrak{A}(t)u = f(t) \qquad \text{in } \Omega \times \dot{J}, \tag{4.9}$$

$$\mathfrak{B}(t)u = g(t) \qquad \text{on } \Gamma \times \dot{J}. \tag{4.10}$$

Definition. A function $u \in C(J, W^{2\beta-2}_{p,\mathfrak{B}}) \cap C^1(\dot{J}, W^{2\beta-2}_{p,\mathfrak{B}})$ satisfying $u(0) = 0$ is called.

- a strong $W^{2\beta}_p$-solution of (4.9)-(4.10) if $2\beta \in (1 + 1/p, 2]$ and if, for each $\varepsilon \in \dot{J}$, u is a strong $W^{2\beta}_p$-solution of:

$$\begin{aligned} \dot{v} + \mathfrak{A}(t)v &= f(t) && \text{in } \Omega \times \dot{J} \cap (\varepsilon, \infty), \\ \mathfrak{B}(t)v &= g(t) && \text{on } \Gamma \times \dot{J} \cap (\varepsilon, \infty), \\ v(\varepsilon) &= u(\varepsilon) && \text{in } \Omega. \end{aligned} \tag{4.11$_\varepsilon$}$$

- a weak $W^{2\beta}_p$-solution of (4.9)–(4.10) if $2\beta \in (1/p, 1 + 1/p)$ and u is a weak $W^{2\beta}_p$-solution of $(4.11)_\varepsilon$ for each $\varepsilon \in \dot{J}$, that is if

$$\langle v, \dot{u}(t)\rangle + \mathbf{a}(t)(v, u(t)) = \langle v, f(t)\rangle + \langle \gamma_\partial v, \delta g(t)\rangle_\partial,$$

for all $t \in \dot{J} \cap (\varepsilon, \infty)$ and all $v \in W^{2-2\gamma}_{p',\mathfrak{B}^\#}$.

Remark 4.8. Every strong $W^{2\beta}_p$-solution is a weak $W^{2\tilde{\beta}}_p$-solution if

$$2\tilde{\beta} \in (1/p, 1 + 1/p).$$

This follows from Green's formulas. Conversely it can also be proved that regular weak $W^{2\tilde{\beta}}_p$-solutions are strong $W^{2\beta}_p$-solutions.

With these definitions we are now able to reformulate the abstract results of the preceding sections.

THEOREM 4.9. Let assumptions (4.6)–(4.8) be fulfilled.

(i) Assume that $2\gamma \in (1/p, 1 + 1/p)$ and that there exists

$$u_g \in C^{\sigma}(\dot{J}, W^{2\gamma}_p) \cap C^{1+\sigma}(\dot{J}, W^{2\gamma-2}_{p,\mathfrak{B}})$$

with $u_g(0) = 0$, $(1 - \delta)u_g = g$ on $\Gamma \times \dot{J}$ and

$$\dot{u}_g(t) + \mathfrak{A}u_g \in C_{k-1}(J, W^{2\gamma-2}_p).$$

Then, if $\sigma > 0$, (4.9)–(4.10) possesses a unique weak $W^{2\gamma}_p$-solution u, and a unique $W^{2\tilde{\gamma}}_p$-solution for $1/p < 2\tilde{\gamma} < 2\gamma$, if $\sigma = 0$. Moreover

$$u \in C^{\nu}(J, W^{2\gamma-2+2\mu}_{p,\mathfrak{B}})$$

for $0 \le \nu < \mu \wedge (1 - \mu) < 1$, provided $\nu \ge 1/k'$.

(ii) Assume that $2\gamma \in (1 + 1/p, 2]$ and that there exists

$$u_g \in C^\sigma(\dot{J}, W_p^{2\gamma}) \cap C^{1+\sigma}(\dot{J}, W_p^{2\gamma-2})$$

with $u_g(0) = 0$, $\mathcal{B}u_g = g$ on $\Gamma \times \dot{J}$ and

$$\dot{u}_g(t) + \mathcal{A}u_g \in C_{k-1}(J, W_p^{2\gamma-2}).$$

Then (4.9)–(4.10) possesses a unique strong $W_p^{2\gamma}$-solution u. Moreover

$$u \in C^\nu(J, W_{p,\mathcal{B}}^{2\gamma-2+2\mu})$$

for $\alpha \in [0, 1)$ and $0 \le \nu < \mu \wedge (1 - \mu) < 1$ such that $2\gamma - 2 + 2\mu < 1 + 1/p$, provided $\nu \ge 1/k'$.

Proof. We divide the proof in two steps corresponding to the two assertions.

(α) We first put

$$E_0 := W_{p,\mathcal{B}}^{2\gamma-2} \qquad E_1 := W_{p,\mathcal{B}}^{2\gamma}.$$

Then we notice that the assumptions on $(\mathcal{A}, \mathcal{B})$ imply by Theorem 4.5 that

$$A_{\gamma-1} \in C_k^\rho(J, \mathcal{H}(E_1, E_0)).$$

If we now define $v_g(t) \in W_{p,\mathcal{B}}^{2\gamma-2}$ by

$$\langle v, v_g(t)\rangle := \mathbf{a}(t)(v, u_g(t)), \qquad t \in J,\ v \in W_{p',\mathcal{B}^\#}^{2-2\gamma},$$

we obtain by the assumptions on A and u_g that

$$v_g \in C^{\rho\wedge\sigma}(\dot{J}, W_{p,\mathcal{B}}^{2\gamma-2}). \tag{4.12}$$

Thus we may reformulate the singular initial boundary value problem in the weak sense. That is we seek a function $v \in C(J, E_0)$ satisfying

$$\langle w, \dot{v}(t)\rangle + \mathbf{a}(t)(w, v(t)) = \langle w, F(t)\rangle$$

for all $t \in \dot{J}$ and all $w \in W_{p',\mathcal{B}^\#}^{2\alpha-2}$. Here we put

$$F := f - \dot{u}_g - v_g + \gamma_{\Gamma_1}' \delta g,$$

Observe that

$$F \in C_{k-1}(J, W_{p,\mathcal{B}}^{2\gamma-2}),$$

since

$$\gamma_{\Gamma_1} \in \mathcal{L}(W_{p,\mathcal{B}^\#}^{2-2\gamma}, W_{p'}^{2-2\gamma-1/p'}(\Gamma_1))$$

and thus

$$\gamma_{\Gamma_1}' \in \mathcal{L}(W_p^{2\gamma-1-1/p}(\Gamma_1), W_{p,\mathcal{B}}^{2\gamma-2}).$$

If we now reinterpret the last equation as an abstract equation in E_0 we are left with the task of finding a solution of

$$\dot{v} + A_{\gamma-1}(t)v = F(t), \qquad t > 0 \tag{4.13}$$

in the space $W^{2\gamma-2}_{p,\mathcal{B}}$ for $F \in C_{k-1}(J, E_0)$, since it is easily seen that $u := v + u_g$ is a solution of the inhomogeneous problem, if v solves (4.13). Thus we may apply Proposition 2.5 to get a solution in the asserted regularity classes. To see that u is a weak $W^{2\gamma}_p$-solution of (4.9)–(4.10) if $\sigma > 0$ and a weak $W^{2\tilde{\gamma}}_p$-solution for $1/p < 2\tilde{\gamma} < 2\gamma$ if $\sigma = 0$, we refer to Theorem 11.2 in [10]. It suffices to apply that theorem to the initial value problem obtained starting at a positive time. To this end property (4.12) is needed.

(β) In this case we first exploit the assumptions to show the existence of a weak solution $\bar{u}$ of (4.9)–(4.10) as in the first part of the proof. Then we make use of the regular theory to handle the following regular problem (for arbitrary $\varepsilon > 0$):

$$\begin{aligned} \dot{u} + \mathcal{A}(t)u &= f && \text{in } \Omega \times \dot{J} \cap (\varepsilon, \infty), \\ \mathcal{B}(t)u &= g(t) && \text{on } \Gamma \times \dot{J} \cap (\varepsilon, \infty), \\ u(\varepsilon) &= \bar{u}(\varepsilon) && \text{in } \Omega \end{aligned}$$

The assertion is then obtained by Theorem 11.3 in [10]. ∎

We turn now to the main result about singular initial boundary value problems: local existence in the quasilinear case. Suppose that

$$2\gamma \in (1/p, 1 + 1/p)$$

and that there exist α and β with $0 < \beta < \alpha < 1$ such that

$$(\mathcal{A}, \mathcal{B}) \in C^{\rho, 1-}_k(J \times W^{2\gamma-2+2\alpha}_{p,\mathcal{B}}, \mathcal{E}^{\varrho}(\Omega)) \tag{4.14}$$

for $2\varrho \in [1, 2]$ with $2\gamma \in [2 - 2\varrho, 2\varrho]$ and

$$(f, g) \in C^{\rho, 1-}_{k-1}(J \times W^{2\gamma-2+2\alpha}_{p,\mathcal{B}}, W^{2\gamma-2+2\beta}_{p,\mathcal{B}} \times \partial W^{2\gamma+2\beta}_p) \tag{4.15}$$

with $(1 - \delta)g = 0$. In addition we assume also that for fixed $t \in J$ and fixed $u \in W^{2\gamma-2+2\alpha}_{p,\mathcal{B}}$ the semigroup generated by $-A(t, u)$ is exponentially decaying at a fixed rate independent of t and u, that is

$$A(t, u) \in \mathcal{H}^{-}(W^{2\gamma}_{p,\mathcal{B}}, W^{2\gamma-2}_{p,\mathcal{B}}). \tag{4.16}$$

Consider the singular parabolic initial boundary value problem

$$\begin{aligned} \dot{u} + \mathcal{A}(t, u)u &= f(t, u) && \text{in } \Omega \times \mathbb{R}_{>0}, \\ \mathcal{B}(t, u)u &= g(t, u) && \text{on } \partial\Omega \times \mathbb{R}_{>0}, \end{aligned}$$

Definition. Let $J \subset \mathbb{R}^+$ be a nontrivial interval containing the origin. A function $u \in C(J, W^{2\gamma-2}_p) \cap C^1(\dot{J}, W^{2\gamma-2}_p)$ is called

- a strong $W^{2\gamma}_p$-solution of the above problem on J if $2\gamma \in (1 + 1/p, 2]$, $u(0) = 0$ and u is, for each $\varepsilon \in \dot{J}$, a strong $W^{2\gamma}_p$-solution of

$$\begin{aligned} \dot{v} + \mathcal{A}(t, u)v &= f(t, u) && \text{in } \Omega \times J \cap (\varepsilon, \infty), \\ \mathcal{B}(t, u)v &= g(t, u) && \text{on } \partial\Omega \times J \cap (\varepsilon, \infty), \\ v(\cdot, \varepsilon) &= u(\cdot, \varepsilon) && \text{in } \Omega. \end{aligned} \tag{4.16}_\varepsilon$$

• a weak $W_p^{2\gamma}$-solution on J if $2\gamma \in (1/p, 1 + 1/p)$, $u(0) = 0$ and u is, for each $\varepsilon \in \dot{J}$, a weak $W_p^{2\gamma}$-solution of $(4.16)_\varepsilon$.

Putting again

$$E_0 := W_{p,\mathcal{B}}^{2\gamma-2} \quad \text{and} \quad E_1 := W_{p,\mathcal{B}}^{2\gamma},$$

we infer from Theorems 7.1 and 7.2 in [10] that

$$E_\theta = (E_0, E_1)_\theta \doteq W_{p,\mathcal{B}}^{2\gamma-2+2\theta}, \tag{4.17}$$

provided the exceptional values are excluded. In particular we have that (without loss of generality) this equality holds for α and β. This means that

$$A := A_{\gamma-1} \in C_k^{\rho,1-}(J \times E_\alpha, \mathcal{H}^-(E_1, E_0)).$$

On the other hand we also see that

$$F := f + \mathfrak{J}_{\gamma+\beta-1} g \in C_{k-1}^{\rho,1-}(J \times E_\alpha, E_\beta),$$

for $2\gamma + 2\beta < 1 + 1/p$, which can be achieved, without loss of generality, by choosing β small enough. Here we put

$$\mathfrak{J}_{\gamma+\beta-1} := \gamma_{\Gamma_1}' \in \mathcal{L}(W_p^{2\gamma+\beta-1-1/p}(\Gamma_1), W_{p,\mathcal{B}}^{2\gamma-2+2\beta})$$

Thus we may rewrite the singular boundary value problem as the following abstract singular Cauchy problem in E_0:

$$\dot{u} + A(t, u)u = F(t, u), \qquad t \in \mathbb{R}_{+0}.$$

We are now able to prove the following theorem.

THEOREM 4.10. Suppose that assumptions (4.14)–(4.15) and (4.16) are satisfied. Then

$$\dot{u} + \mathcal{A}(t, u)u = f(t, u) \qquad \text{in } \Omega \times \mathbb{R}_{>0},$$

$$\mathcal{B}(t, u)u = g(t, u) \qquad \text{on } \partial\Omega \times \mathbb{R}_{>0},$$

possesses a unique maximal solution $u \in C(J^+, E_\alpha)$, where J^+ is the maximal interval of existence. It may be recovered as the unique fixed point of the following abstract singular variation-of-constants-formula in the Banach space $W_{p,\mathcal{B}}^{2\gamma-2}$:

$$u(t) = \int_0^t U_{A(\cdot,u)}(t, \tau) F(\tau, u(\tau))\, d\tau.$$

Furthermore $u \in C^\nu(J^+, E_\mu) \cap C(\dot{J}^+, E_1) \cap C^1(\dot{J}^+, E_0)$ for each $0 \le \nu < \mu \wedge (1 - \mu) < 1$, provided $\nu \ge 1/k'$. Lastly u is a weak $W_p^{2\gamma}$-solution of the singular initial boundary value problem.

Proof. By means of the Dirichlet form $\mathbf{a}(t, u)$ associated to $(\mathcal{A}, \mathcal{B})(t, u)$ we rewrite the equations under consideration as the following equation

$$\langle \dot{u}, v\rangle + \mathbf{a}(t, u)(u, v) = \langle f(t, u), v\rangle + \langle \gamma_{\Gamma_1} v, g(t, u)\rangle,$$

which has to be satisfied by all t and all test-functions $v \in W^{2-2\gamma}_{p',\mathcal{B}^\#}$. This equation may, in its turn, be interpreted as an abstract singular evolution equation in the Banach space E_0, observing that

$$\langle \gamma_{\Gamma_1} v, g(t, u)\rangle = \langle \gamma'_{\Gamma_1} g(t, u), v\rangle = \langle \mathfrak{J}_{\gamma+\beta-1} g(t, u), v\rangle.$$

This yields namely

$$\dot{u} + A(t, u)u = F(t, u) := \langle f(t, u), \cdot\rangle + \mathfrak{J}_{\gamma+\beta-1} g(t, u).$$

Assumption (4.15) implies now that

$$[(t, u) \mapsto F(t, u)] \in C^{\rho, 1-}_{k-1}(J \times E_\alpha, E_\beta),$$

where we also used (4.17). On the other hand , as we already saw, it is a consequence of Theorem 4.5, of the interpolation properties of the spaces under consideration and of the hypotheses concerning A that

$$a \in C^{\rho, 1-}_{k}(J \times E_\alpha, \mathcal{H}^-(E_1, E_0)).$$

Theorem 3 together with Theorem 13.1 in [10] implies now the existence of a unique maximal solution

$$u \in C^\nu(J^+, E_\mu) \cap C(\dot{J}^+, E_1) \cap C^1(\dot{J}^+, E_0),$$

for ν and μ as in the assertion. Hence we argue that $G(\cdot) := F(\cdot, u(\cdot))$ satisfies

$$G \in C^\varrho(\dot{J}^+, E_\beta),$$

and that

$$A(\cdot, u(\cdot)) \in C^\varrho(\dot{J}^+, \mathcal{H}^-(E_1, E_0)),$$

for some $\varrho \in (0, 1)$, since u is the above Hölder regularity classes. From this we deduce that u, as a solution of

$$\dot{v} + A(t, u(t))v = G(t), \qquad t > 0$$

in the Banach space E_0, is weak $W^{2\gamma}_p$-solution on J^+ of the singular quasilinear initial boundary value problem by Theorem 4.9. ■

Remark 4.11. We observe that if one is interested in the existence of strong solution for the quasilinear singular initial boundary value problem then one may proceed as in the linear case. That is, first establishing the existence of weak solutions and then arguing by means of the regular theory of initial boundary value problems that the solutions have better regularity. Correspondingly stronger hypotheses about the data are of course needed. For a more detailed treatment we refer to [10], in particular to paragraph 13 therein.

Remark 4.12. Following the way sketched in the previous remark and making use of some bootstrapping arguments, it is possible to show the existence of smooth solutions for smooth data (cf. for instance [10, paragraph 14, in particular the remarks preceding Theorem 14.6]).

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