Calc. Var. (2017) 56:85
DOI 10.1007/s00526-017-1181-1

Calculus of Variations

Domain variations and moving boundary problems

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Received: 8 August 2016 / Accepted: 4 May 2017 / Published online: 2 June 2017

Abstract In the past few decades maximal regularity theory has successfully been applied to moving boundary problems. The basic idea is to reduce the system with varying domains to one in a fixed domain. This is done by a transformation, the so-called Hanzawa transformation, and yields a typically nonlocal and nonlinear coupled system of (evolution) equations. Well-posedness results can then often be established as soon as it is proved that the relevant linearization is the generator of an analytic semigroup or admits maximal regularity. To implement this program, it is necessary to somehow parametrize to space of boundaries/domains (typically the space of compact hypersurfaces Γ in $\mathbb{R}^n$, in the Euclidean setting). This has traditionally been achieved by means of the already mentioned Hanzawa transformation. The approach, while successful, requires the introduction of a smooth manifold Γ_∞ close to the manifold Γ_0 in which one cares to linearize. This prevents one to use coordinates in which Γ_0 lies at their "center". As a result formulæ tend to contain terms that would otherwise not be present were one able to linearize in a neighborhood emanating from Γ_0 instead of from Γ_∞. In this paper it is made use of flows (curves of diffeomorphisms) to obtain a general form of the relevant linearization in combination with an alternative coordinatization of the manifold of hypersurfaces, which circumvents the need for the introduction of a "phantom" reference manifold Γ_∞ by, in its place, making use of a "phantom geometry" on Γ_0. The upshot is a clear insight into the structure of the linearization, simplified calculations, and simpler formulæ for the resulting linear operators, which are useful in applications.

Mathematics Subject Classification 35A05 · 35A07 · 35K35 · 35R35

Communicated by Y. Giga.

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1 Introduction

Moving boundary problems are ubiquitous and numerous in applications. Latter include, but are by no means limited to, fluid dynamics with, e.g. the classical Stefan problem, and biology with, e.g. models of tumor growth.

In abstract terms such problems consist of a system of (initial) boundary value problems for unknown physical quantities (read concentrations, velocity fields, temperature,...) and for at least one unknown (evolving) domain in which the boundary value problems are set. Even when the equations for the unknown physical quantities appear linear, the system is not, due to the coupling with the geometry. Indeed, two solutions living on two distinct domains can not be added to obtain a new solution on a new domain.

A versatile general purpose approach to (fully) nonlinear evolution equations of parabolic type is given by *optimal (also maximal) regularity theory*, see e.g. [3,4,17]. In a nutshell, the approach consists in linearizing a nonlinear equation/system in a point in the space of unknowns, prove that the linearization is an isomorphism (between carefully chosen function spaces), and eventually solving the equations by a perturbation argument.

In the context of free and moving boundary problems, linearization in the unknown necessarily includes taking domain variations (recall that the domain is itself an unknown of the problem). This amounts to measuring the infinitesimal dependence of functions, differential operators, pseudo-differential operators, and geometric quantities on the domains on which they are defined.

To be more specific consider a domain Ω_0 in $\mathbb{R}^n$, $n \in \mathbb{N}$, defined by its boundary Γ_0 as the bounded region inside of it. It is supposed that Γ_0 be a compact hypersurface of limited regularity, say C^2, for now. For technical reasons that will become more explicit shortly, the parametrization problem has traditionally been solved by introducing coordinates in a neighborhood of Γ_0 (it is clearly enough to vary the boundary as a means to vary the domain) based on a smooth (C^∞ or analytic) manifold Γ_∞ arbitrarily close to Γ_0 (in the C^2 sense). The basic idea consists in parametrizing the surface Γ_0 over Γ_∞ as a graph in "normal direction", that is, by a function

$$\rho_0 : \Gamma_0 \to \mathbb{R}$$

via

$$\Gamma_0 = \big\{ y + \rho_0(y)\nu_\infty(y) \,\big|\, y \in \Gamma_\infty \big\},$$

where ν_∞ is the (smooth) outward unit normal to Γ_∞. In this way, the unknown domain can be described (locally in time, but this is enough) as an unknown function and the geometry (read ν_∞) does not impose any limitations since it is taken to be smooth. Notice that it would be impossible to choose Γ_0 as a reference manifold since it would require one to use its unit outward normal field ν_0, which enjoys one less degree of regularity as compared to the manifold itself. As shall become evident later, this loss of regularity cannot be afforded if one is to take the optimal regularity approach briefly sketched above. This is the core idea of the transformation, which was first employed for the Stefan problem [13], and that has become known as the Hanzawa transformation. This approach has been used repeatedly and was nicely expounded in [19].

It is the purpose of this paper to overcome the "regularity issue" in an alternative way that does not require the use of a smooth reference manifold, but rather uses the surface Γ_0 as the center of the coordinate patch, in which, after all, the linearization is needed. The idea can be simply stated: instead of using a smooth "phantom manifold" Γ_∞, introduce a regularized normal field ν_0^δ on Γ_0 and use it to parametrize a neighborhood of Γ_0 in the space of surfaces. This can be thought of as using a "phantom geometry" on Γ_0. Notice that in the smooth case, the two approaches coincide, since Γ_0 with its natural geometry can always be chosen as reference manifold. An additional goal of this paper is to offer a more geometrical approach to the issue of linearization. It uses flows and, more in general, curves of diffeomorphisms to conveniently identify it. This results in simpler and more transparent calculations which can be performed before the parametrization described above for the unkown surface is introduced, at the very end, in order to obtain a system of PDEs for unknown functions only.

An added advantage of the approach is the simplified form taken by the linearization which significantly shortens the analysis required to prove that it is a generator of an analytic semigroup, enjoys maximal regularity, or to obtain spectral information for stability analysis. A prototypical example is discussed at the end of the paper.

It should be noticed that, for some equations, an alternative method is available which allows one to avoid working on the manifold of hypersurfaces. The basic idea consists in replacing the unknown surface $\Gamma(t)$ by a function of which it is the level set of, as is, for instance, the associated (signed) distance function $d(t) = d(\cdot, \Gamma(t))$ and derive equations for it (at least locally). This method can be traced back to [7,9,10]. See also [17, Section 8.5.4] for a concise description. While this approach produces equations in Euclidean space, this is only so if the ambient space is itself Euclidean. The same remark applies to the use of smooth reference manifolds described above, which, in a non Euclidean context, could only be as smooth as the ambient manifold. The approach advocated here has the advantage to work, with the appropriate modifications, just as well if the ambient space is a manifold, even of finite degree of smoothness.

While the natural idea of taking domain variations in order to measure dependence of various quantities of physical interest, such as boundary value problems, on the domain on which they are defined has been implicitly utilized in the literature before in various contexts [1,12,14,18,20], this paper offers, in this respect, a simple and streamlined procedure and presentation which makes the geometric nature of the problem fully transparent.

2 Preliminaries

Basic facts from differential geometry and manifold theory will be used freely in the sequel. It is referred to standard references such as [15,16,21] for the required background. For $\alpha \in (0, 1)$ and $k \in \mathbb{N}$, denote by $\mathcal{M}^{k+\alpha}$ the space of embedded hypersurfaces of $\mathbb{R}^n$ given by

$$\mathcal{M}^{k+\alpha} = \Big\{ \Gamma \subset \mathbb{R}^n \,\Big|\, \Gamma \text{ compact, connected, orientable hypersurface of class } buc^{k+\alpha} \Big\}, \tag{2.1}$$

where the regularity space $buc^{k+\alpha}$ is the so-called little Hölder space. For an open subset $O \subset \mathbb{R}^n$, the latter is defined as the closure of the regular space of bounded and uniformly Hölder continuous functions given by

$$\mathrm{BUC}^{k+\beta}(O) = \Big\{ f : O \to \mathbb{R} \,\Big|\, f \in \mathrm{BUC}^k(O) \quad \text{and} \quad \partial^\gamma f \in \mathrm{BUC}^\beta(O) \quad \text{for} \quad |\gamma| = k \Big\},$$

with $\beta > \alpha$, in the topology determined by the norm $\| \cdot \|_{k+\alpha,\infty}$ defined through

$$\|f\|_{k+\alpha,\infty} = \max_{|\gamma| \le k} \|\partial^\gamma f\|_\infty + \max_{|\gamma| = k} [\partial^\gamma f]_\alpha,$$

where

$$[g]_\alpha = \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\alpha},\ g \in \mathrm{C}(O).$$

If O is replaced by a compact manifold $M \in buc^{k+\alpha}$, then the spaces $buc^{l+\beta}(M)$, for $l \leq k$ and $\beta \in (0, 1)$ with $\beta \leq \alpha$ if $l = k$, are defined in the standard way by resorting to localizations combined with a smooth partition of unity.

Remark 2.1 The choice of little Hölder spaces is motivated by the ease provided by the use of a family of function spaces densely embedded in one another in the context of maximal regularity for generators of analytic semigroups. The space $buc^\alpha(M)$ consists of those $\mathrm{BUC}^\alpha(M)$ functions g satisfying

$$\lim_{\delta \to 0} \sup_{y \neq z \in \mathbb{B}_M(x,\delta)} \frac{|g(y) - g(z)|}{d_M(y, z)^\alpha} = 0.$$

For all considerations preceding the final example, they can be replaced by the more standard classes of Hölder regularity $BUC^{k+\alpha}$, of which they are closed subspaces.

While this choice of spaces is not essential until maximal regularity results are used and, even then not unique, it is made for consistency with the final part of the paper and for simplicity of presentation. It is referred to [5] and the references cited therein for alternative functional settings in which maximal regularity holds. As the preferred spaces are a matter of taste and not of necessity in most applications, the choice made here is not restrictive but allows for a more concise presentation.

Given $\Gamma_0 \in \mathcal{M}^{2+\alpha}$, a little room is needed in which to operate. It is provided by the following lemma.

Lemma 2.2 (Existence of a tubular neighborhood) *Given* $\Gamma_0 \in \mathcal{M}^{2+\alpha}$*, there is* $r_0 > 0$ *such that*

$$\mathcal{T}_{r_0}(\Gamma_0) := \{x \in \mathbb{R}^n \mid |d(x, \Gamma_0)| < r_0\}$$

is an open neighborhood of Γ_0 *diffeomorphic to* $\Gamma_0 \times (-r_0, r_0)$.

Notice that $d(\cdot, \Gamma_0)$ will always denote the signed distance to Γ_0 with the understanding that it is negative in the interior of the domain bounded by Γ_0.

Proof While the proof is well-known, it is given anyway as a way to introduce some notation which will be useful again later.

By assumption Γ_0 has bounded principal curvatures. Fix a point $y \in \Gamma_0$ and choose coordinates $\sigma = \left(\sigma^1, \dots, \sigma^{n-1}\right)$ such that $\tau_j^0 = \frac{\partial}{\partial \sigma^j}$, $j = 1, \dots, n-1$, is a orthonormal basis of $T_y\Gamma_0$ consisting of principal directions, i.e. satisfying

$$d_{\tau_j^0}\nu_0 = \frac{d}{d\sigma^j}\Big|_{\sigma=0}\nu_0 = \lambda_j^0\tau_j^0 \quad \text{for} \quad j = 1, \dots, n-1,$$

where $\nu_0 = \nu_{\Gamma_0}$ and the dependence on y or σ is omitted, and λ_j^0 are the principal curvatures of Γ_0 at y. It is assumed that $\tau_1^0, \dots, \tau_{n-1}^0, \nu_0$ is a positively oriented orthonormal basis of $T_y\mathbb{R}^n$. Define the map

$$\Phi : \Gamma_0 \times (-r_0, r_0) \to \mathbb{R}^n,\ (y, r) \mapsto y + r\nu_0(y),$$

and notice that $\Phi \in buc^{1+\alpha}\big(\Gamma_0 \times (-r_0, r_0)\big)$. It follows from the choice of coordinates that

$$\frac{\partial \Phi}{\partial \sigma^j} = \tau_j^0 + r d_{\tau_j^0} \nu = \left(1 + r\lambda_j^0\right) \tau_j^0,$$
$$\frac{\partial \Phi}{\partial r} = \nu_0.$$

Then one has that

$$\frac{\partial \Phi}{\partial \sigma^j} \cdot \frac{\partial \Phi}{\partial \sigma^k} = \delta_{jk} \left(1 + r\lambda_j^0\right)^2, \; j, k = 1, \dots, n-1,$$
$$\frac{\partial \Phi}{\partial \sigma^j} \cdot \frac{\partial \Phi}{\partial r} = 0, \; j = 1, \dots, n-1,$$
$$\frac{\partial \Phi}{\partial r} \cdot \frac{\partial \Phi}{\partial r} = 1.$$

By assumption

$$\max_{j=1,\dots,n-1} \left|\lambda_j^0\right| \leq \Lambda < \infty \quad \text{on} \quad \Gamma_0,$$

and, consequently, $D\Phi(y, r)$ is invertible for $0 \leq r < \tilde{r}_0$ and some $\tilde{r}_0 > 0$ which is taken to coincide with r_0 without loss of generality. This holds independently of the point $y \in \Gamma_0$. Compactness and the inverse function theorem then imply that

$$\Phi\big|_{\mathbb{B}_{\Gamma_0}(y_j, r_0) \times (-r_0, r_0)}$$

is a diffeomorphism onto its image and

$$\cup_{l=1,\dots,N} \mathbb{B}_{\Gamma_0}(y_l, r_0) \supset \Gamma_0,$$

for some $N \in \mathbb{N}$ and some $y_l \in \Gamma_0$, $l = 1, \dots, N$. It remains to make sure that the hypersurface does not come close to itself (not in a local fashion, but rather in a global way) in order to obtain a global diffeomorphism. To that end, define

$$\delta_l = \inf_{y \in \mathbb{B}_{\Gamma_0}(y_l, r_0)^{\mathsf{c}}} d_{\mathbb{R}^n}(y, y_l)$$

and reset r_0 to half of $\delta = \min_{l=1,\dots,N} \delta_l$. Then $\Phi\big|_{\Gamma_0 \times (-r_0, r_0)}$ is injective as desired. Indeed, if

$$\Phi(y_1, r_1) = \Phi(y_2, r_2) = x \quad \text{for} \quad (y_i, r_i) \in \Gamma_0 \times (-r_0, r_0), \; i = 1, 2,$$

then

$$d_{\mathbb{R}^n}(y_1, y_2) \leq d_{\mathbb{R}^n}(y_1, x) + d_{\mathbb{R}^n}(x, y_2) < \delta,$$

so that y_1, y_2 must be in the same ball and thus coincide along with $r_1 = r_2$. □

Remark 2.3 Observe that the above construction yields a foliation of the tubular neighborhood by $buc^{1+\alpha}$ surfaces only, since it employs the normal of Γ_0.

Remark 2.4 The map Φ defined in the above proof yields coordinates (y, r) in $\mathcal{T}_{r_0}(\Gamma_0)$. In these variables it holds that $d((r, y), \Gamma_0) = r$ for the signed distance function to Γ_0. It readily follows that

$$\nabla d(\cdot, \Gamma_0) = 1 \, \frac{\partial}{\partial r} = \nu_0.$$

It can, in fact, be shown (see e.g. [11, Section 14.6]) that $d(\cdot,\Gamma_0)\in buc^{2+\alpha}\big(\mathcal{T}_{r_0}(\Gamma_0)\big)$ and that

$$\Delta d(\cdot,\Gamma_0)=H_{\Gamma_0},$$

on Γ_0, where H is the mean curvature of Γ_0.

The next lemma gives a refined version of the above one which preserves regularity.

Lemma 2.5 *Given $\Gamma_0\in\mathcal{M}^{2+\alpha}$, there is $r_0>0$ and hypersurfaces $\Gamma_r\in\mathcal{M}^{2+\alpha}$ for $r\in(-r_0,r_0)$ such that*

$$\bigcup_{|r|<r_0}\Gamma_r$$

is an open neighborhood of Γ_0.

Proof By Lemma 2.2 there is $\tilde r_0>0$ such that, given any $x\in\mathcal{T}_{\tilde r_0}(\Gamma_0)$, there is one

$$(y,r)=\big(y(x),r(x)\big)\quad\text{s.t.}\quad x=y+r\nu_{\Gamma_0}(y).$$

In $\mathcal{T}_{\tilde r_0}(\Gamma_0)$ define the field

$$\tilde{\tilde{\nu}}(x)=\nu_{\Gamma_0}\big(y(x)\big),$$

take a smooth cut-off function $\eta:\mathbb{R}\to\mathbb{R}$ with

$$0\le\eta\le 1,\ \eta|_{[-\tilde r_0/2,\tilde r_0/2]}\equiv 1,\quad\text{and}\quad \eta|_{(-3\tilde r_0/4,3\tilde r_0/4)^{\mathsf{c}}}\equiv 0,$$

and set

$$\widetilde{\nu}(x)=\begin{cases}\tilde{\tilde{\nu}}(x)\eta\big(r(x)\big), & x\in\mathcal{T}_{\tilde r_0}(\Gamma_0),\\ 0, & x\notin\mathcal{T}_{\tilde r_0}(\Gamma_0).\end{cases}$$

Then $\widetilde{\nu}\in buc^{1+\alpha}(\mathbb{R}^n,\mathbb{R}^n)$ is a global vector field. Now take a compactly supported smooth mollifier ψ_δ and define

$$\nu^\delta=\psi_\delta*\widetilde{\nu}$$

componentwise. It follows that $\nu^\delta\in BUC^\infty(\mathbb{R}^n,\mathbb{R}^n)$, that $\operatorname{supp}(\nu^\delta)\subset\mathcal{T}_{\tilde r_0}(\Gamma_0)$ for small $\delta>0$, and that

$$\nu^\delta\to\nu\big(y(\cdot)\big)\quad\text{in}\quad buc^{1+\alpha}\big(\mathcal{T}_{\tilde r_0/2}(\Gamma_0)\big)\quad\text{as}\quad\delta\to 0.$$

In particular

$$|\nu^\delta(y)\cdot\tau_{\Gamma_0}(y)|\le c(\delta),\ \forall\tau_{\Gamma_0}(y)\in\mathcal{T}_y\Gamma_0\quad\text{with}\quad|\tau_{\Gamma_0}(y)|=1,$$

where $c(\delta)\to 0$, as δ tends to zero, uniformly in $y\in\Gamma_0$. The vector field is therefore uniformly transversal to Γ_0. Finally set

$$\Gamma_r=\varphi^\delta(\Gamma_0,r)$$

for the flow generated by the ODE

$$\begin{cases}\dot x=\nu^\delta(x),\\ x(0)=y\in\Gamma_0.\end{cases}$$

It is easily seen that there is $r_0 > 0$ such that

$$\varphi^\delta(\Gamma_0, r) = \Gamma_r \subset \mathcal{T}_{\tilde{r}_0}(\Gamma_0),\ |r| \leq r_0,$$

if $\delta << 1$, and standard ODE arguments yield that

$$\varphi^\delta : \Gamma_0 \times (-r_0, r_0) \to \bigcup_{r \in (-r_0, r_0)} \Gamma_r$$

is a diffeomorphism. □

The previous lemma provides coordinates (y, r) for a neighborhood of Γ_0, which can be denoted by $\mathcal{T}^{\nu^\delta}_{r_0}(\Gamma_0)$ since it is constructed starting with the smooth vector field ν^δ. Explicitly this means that

$$\forall\, x \in \mathcal{T}^{\nu^\delta}_{r_0}(\Gamma_0)\ \exists!\, (y, r) \quad \text{s.t.} \quad x = \varphi^\delta(y, r).$$

The next lemma uses the Hausdorff distance on compact subsets, which is given by

$$d_{\mathrm{C}^0}(K, \widetilde{K}) = \max\Big\{ \max_{\overline{x} \in \overline{K}} d(\overline{x}, K), \max_{x \in K} d(x, \widetilde{K}) \Big\}$$

to define a distance d_{C^1} between $\Gamma, \widetilde{\Gamma} \in \mathcal{M}^{1+\alpha}$ in the following manner

$$d_{\mathrm{C}^1}(\Gamma, \widetilde{\Gamma}) = d_{\mathrm{C}^0}(N\Gamma, N\widetilde{\Gamma}) \tag{2.2}$$

where

$$N\Gamma = \big\{ \big(y, \nu_\Gamma(y)\big) \,\big|\, y \in \Gamma \big\} \subset \mathbb{R}^{2n},$$

where $\nu_\Gamma(y)$ denotes the unit, outward pointing normal to Γ at y. Proximity in d_{C^1} therefore implies not only that the hypersurfaces are close to each other but that also their tangent spaces cross everywhere at uniformly small angles. This is used to exclude "rough" (oscillatory) approximations.

Lemma 2.6 *Let $\Gamma_0 \in \mathcal{M}^{j+\alpha}$, $j \geq 2$, and $\Gamma \in \mathcal{M}^{k+\alpha}$, $j \geq k \geq 1$, satisfy $d_{\mathrm{C}^1}(\Gamma, \Gamma_0) << 1$. Then there is a unique $\rho \in buc^{k+\alpha}(\Gamma_0)$ such that*

$$\Gamma = \big\{ \varphi^\delta\big(y, \rho(y)\big) \,\big|\, y \in \Gamma_0 \big\}.$$

In more suggestive terms, Γ can be viewed as a ν^δ-graph (or a φ^δ-graph) over Γ_0.

Proof Without loss of generality assume that $j = 2$ and let $k = 2$ first. Given Γ with the above properties, it immediately follows from Lemma 2.5 that, given any $x \in \Gamma$, there is a unique $\big(y(x), r(x)\big) \in \Gamma_0 \times (-r_0, r_0)$ such that

$$x = \varphi^\delta\big(y(x), r(x)\big).$$

Now, by assumption, one has that $|\nu_\Gamma(x) - \nu_0(\tilde{y})| << 1$ for some $\tilde{y} \in \Gamma_0$ for which $|x - \tilde{y}| << 1$. Since $|\nu_0(\tilde{y}) - \nu^\delta(\tilde{y}, r)| << 1$ uniformly in $\tilde{y}$ and in $r \in [-r_0, r_0]$ for some $r_0 > 0$, it follows that φ^δ flows transversal (in fact almost orthogonal) to both Γ_0 and Γ. Take $\bar{y} \in \Gamma_0$, then $\varphi^\delta(\bar{y}, \cdot)$ must cross Γ as the flow φ^δ reaches the boundaries of the tubular neighborhood $\mathcal{T}^\delta_{r_0}(\Gamma_0)$ containing Γ and Γ splits $\mathbb{R}^n$ in two parts, each containing one of the connected components of $\partial\mathcal{T}^\delta_{r_0}(\Gamma_0)$. The number $N = N(\bar{y})$ of crossings must be finite due to the transversality of φ^δ and to the smoothness of Γ. Let $\bar{r}_1 < \cdots < \bar{r}_N$ be the crossings "over" $\bar{y}$ and consider the map d^δ_Γ that takes (y, r) to $d\big(\varphi^\delta(y, r), \Gamma\big)$. The latter is defined in at least a neighborhood $\mathcal{U}_j$ of $(\bar{y}, \bar{r}_j)$ for $j = 1, \dots, N$ due to the smoothness of Γ and satisfies $d_\Gamma^\delta(\bar{y}, \bar{r}_j) = 0$. Observing that

$$\frac{\partial}{\partial r} d_\Gamma^\delta(y, r) = \nabla d\big(\varphi^\delta(y, r), \Gamma\big) \cdot \dot{\varphi}^\delta(y, r),$$

where the dot means differentiation with respect to r, the assumption implies that

$$\nabla d\big(\varphi^\delta(y, r), \Gamma\big) = \nu_\Gamma(x) \simeq \nu_0(y)$$

if $\varphi^\delta(r, y) = x \in \Gamma$, and thanks to $\nabla d(\cdot, \Gamma) = \nu_\Gamma$ on Γ (see Remark 2.4). Simultaneously

$$\dot{\varphi}^\delta(y, r) = \nu^\delta\big(\varphi^\delta(y, r)\big) \simeq \nu_0(y).$$

It follows that $\frac{\partial}{\partial r} d_\Gamma^\delta(\bar{y}, \bar{r}_j) \neq 0$ where $\varphi^\delta(\bar{y}, \bar{r}_j) \in \Gamma$ and $j = 1, \dots, N$. The implicit function theorem now implies the existence of

$$\rho_j \in buc^{k+\alpha}(\mathcal{U}_{0,j}) \quad \text{s.t.} \quad d_\Gamma\big(\varphi^\delta(y, \rho_j(y))\big) \equiv 0, \;\; y \in \mathcal{U}_{0,j},$$

where $\mathcal{U}_{0,j}$ is a neighborhood of $\bar{y}$ in Γ_0 and $j = 1, \dots, N$. Thus Γ can be represented as a graph over Γ_0 locally around the points $(\bar{y}, \bar{r}_j)$ for $j = 1, \dots, N$. In particular, the function N is locally constant, and, thanks to the fact that Γ_0 is connected, actually constant. Now, if $N \geq 2$, then

$$\rho_2(y) - \rho_1(y) > 0 \quad \text{for all} \quad y \in \Gamma_0,$$

and compactness of Γ_0 combined with the continuity of ρ_j, $j = 1, 2$, yields

$$\min_{y \in \Gamma_0} \big[\rho_2(y) - \rho_1(y)\big] > 0.$$

Consequently Γ cannot be connected and this contradiction implies that $N = 1$. As pointed out at the begining of this proof, Lemma 2.5 ensures the existence of $\big(y(x), r(x)\big) \in \mathcal{T}_{r_0}^\delta(\Gamma_0)$ with $x = \varphi^\delta\big(y(x), r(x)\big)$ and shows that the map

$$\Gamma_0 \ni y \mapsto \rho_1(y) = \rho(y) \in \Gamma$$

is onto.

Assume next that $k = 1$. Approximate Γ in $buc^{1+\alpha}$ by a family of $buc^{2+\alpha}$ hypersurfaces $(\Gamma^\eta)_{\eta \in (0,1]}$. This can be done, as for instance in [6], by solving

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega, \end{cases}$$

and setting $\Gamma^\eta = [u = \eta]$. For each $\eta \in (0, 1]$, by the first part of the proof, there is a function $\rho^\eta \in buc^{2+\alpha}(\Gamma_0)$ with the property that

$$\Gamma^\eta = \big(\varphi^\delta \circ (\mathrm{id}, \rho^\eta)\big)(\Gamma_0).$$

Now $d_{C^1}(\Gamma, \Gamma_0) << 1$, the fact that $\|\rho^\eta\|_\infty < \infty$ uniformly in η, and the C^1 convergence of Γ^η to Γ implies that necessarily

$$\|\rho^\eta\|_\infty + \|d\rho^\eta\|_\infty \leq c < \infty \quad \text{for} \quad \eta \in (0, 1],$$

for some positive constant c. If this were not the case, then at least one of the tangent vectors

$$\widetilde{\tau}_j^\eta = d\big[\varphi^\delta \circ (\mathrm{id}, \rho^\eta)\big]\big(\tau_j^0\big) = (d\varphi^\delta) \circ (\mathrm{id}, \rho^\eta)(\tau_j^0) + \big[\nu^\delta \circ (\mathrm{id}, \rho^\eta)\big] d\rho^\eta\big(\tau_j^0\big), \tag{2.3}$$

where τ^0_j, $j=1,\dots,n$, is a basis of $T\Gamma_0$, would eventually point, somewhere, in direction of $\nu^\delta \simeq \nu_0$. This follows from the fact that the first summand in the right-hand-side of (2.3) remains bounded by construction, while $\partial_j \rho^\eta = d\rho^\eta(\tau^0_j)$ would, for at least one j, tend to infinity in size along a sequence $(y_k)_{k\in\mathbb{N}}$ of points in Γ_0. As this sequence can be taken to converge to a point on Γ_0 without loss of generality, a contradiction would ensue to the assumption that $d_{\mathrm{C}^1}(\Gamma,\Gamma_0) << 1$. The Arzéla–Ascoli theorem then implies the existence of a continuous limiting function $\rho_\Gamma : \Gamma_0 \to \mathbb{R}$ such that $\rho^{\eta_k} \to \rho_\Gamma$, as $k\to\infty$, for a sequence of indices $(\eta_k)_{k\in\mathbb{N}}$. It must then hold that

$$\Gamma = \big(\varphi^\delta \circ (\mathrm{id}, \rho_\Gamma)\big)(\Gamma_0)$$

and that $\rho \in buc^{1+\alpha}(\Gamma_0)$ due to the regularity of φ^δ and that of Γ itself. More explicitly, notice that one necessarily has that

$$d\big(\varphi^\delta \circ (\mathrm{id}, \rho_\Gamma)\big) \cdot \nu_\Gamma \circ (\mathrm{id}, \rho_\Gamma) \equiv 0,$$

and, consequently that

$$\big(\nu^\delta \cdot \nu_\Gamma\big) \circ (\mathrm{id}, \rho_\Gamma) d\rho_\Gamma = -d_y \varphi^\delta \cdot \nu_\Gamma \circ (\mathrm{id}, \rho_\Gamma).$$

Since $\nu^\delta\big|_\Gamma \cdot \nu_\Gamma \neq 0$ by assumption, the claimed regularity is implied by the formula obtained for $d\rho_\Gamma$ by dividing through by that non-zero term. □

The above lemma shows that, given $\rho \in buc^{2+\alpha}$ small enough (in the C^1 topology), the hypersurface

$$\Gamma_\rho = \big\{\varphi^\delta\big(y, \rho(y)\big) \,\big|\, y \in \Gamma_0\big\}$$

is well-defined.

It is important to have access to relevant geometric quantities for Γ_ρ. Fix $y \in \Gamma_0$ and choose again coordinates $\sigma = (\sigma^1, \dots, \sigma^{n-1})$ along the principal directions of Γ_0 at y (just as in the proof of Lemma 2.2 and using the notation introduced there). One computes that

$$\tilde{\tau}^\rho_j = \frac{\partial}{\partial \sigma^j} \varphi^\delta \circ (\mathrm{id}, \rho) = \partial_j \varphi^\delta \circ (\mathrm{id}, \rho) + \dot{\varphi}^\delta \circ (\mathrm{id}, \rho) \partial_j \rho \tag{2.4}$$

is a tangent vector to Γ_ρ at $\varphi^\delta\big(y, \rho(y)\big)$. Observe that the notation

$$\partial_j g(y) = \Big\langle d_y g, \tau^0_j \Big\rangle,\ \ y \in \Gamma_0$$

was used in the above expressions for functions defined on Γ_0. For $\delta << 1$ one has that

$$\frac{\partial}{\partial \sigma^j} \varphi^\delta \circ (\mathrm{id}, \rho) \simeq \Big(1 + \lambda^0_j \rho\Big)\, \tau^0_j + (\partial_j \rho)\, \nu_0,$$

since

$$\varphi^\delta(y, r) \simeq y + r\nu^\delta(y) \simeq y + r\nu_0(y).$$

It can be concluded that

$$\tilde{\tau}^\rho_1, \dots, \tilde{\tau}^\rho_{n-1}$$

is a basis of $T_x\Gamma_\rho$ for $x = \varphi^\delta\big(y, \rho(y)\big)$, provided that, as it is assumed, ρ is small in the C^1 topology.

3 Taking variations by flows

Of interest is the dependence of various quantities on the domain/manifold on which they are defined. Fix a compact oriented hypersurface $\Gamma_0 \in \mathcal{M}^{2+\alpha}$ and, for now, let F be any smooth section of a bundle over $\mathcal{M}^{2+\alpha}$, which, in fact, can be assumed to be defined in a neighborhood $\mathcal{U}^{2+\alpha}$ of Γ_0 only. In particular it will be useful to have a convenient way to compute $\frac{d}{d\Gamma}\big|_{\Gamma=\Gamma_0} F$. A natural way to do this is to fix a C^∞-flow φ on $\mathbb{R}^n$, that is, a smooth map

$$\varphi : (-\varepsilon, \varepsilon) \times \mathbb{R}^n \to \mathbb{R}^n, (s, x) \mapsto \varphi(s, x) =: \varphi_s(x),$$

with $\varphi_s \in \mathrm{Diff}^\infty(\mathbb{R}^n)$ for $s \in (-\varepsilon, \varepsilon)$ and satisfying

$$\begin{cases} \varphi_0 = \mathrm{id}_{\mathbb{R}^n}, \\ \varphi_{s+\tilde{s}} = \varphi_s \circ \varphi_{\tilde{s}},\ s, \tilde{s}, s+\tilde{s} \in (-\varepsilon, \varepsilon). \end{cases}$$

and use it in order to generate a curve of hypersurfaces in $\mathcal{U}^{2+\alpha}$ by setting

$$\Gamma_s = \varphi_s(\Gamma_0),\ s \in (-\varepsilon, \varepsilon).$$

Then

$$d_{\Gamma_0} F\big([\Gamma_\cdot]\big) = \big[F \circ \Gamma_\cdot\big] = \big[\big(\varphi_\cdot^* F\big)(\Gamma_0)\big],$$

where the superscript $*$ denotes the pull-back and the square brackets are used to indicate the equivalence class of curves determined by the curve they contain. Proceeding in this way, it is natural to identify the tangent vector $[\Gamma_\cdot]$ with the vector field

$$\frac{d}{ds}\Big|_{s=0} \varphi_s = \nu_\varphi$$

associated to the flow φ.

Remark 3.1 Only the values of ν_φ on Γ_0 actually matter but it is convenient to think of the vector field being defined in at least a neighborhood of Γ_0 and sometimes everywhere. Observe that different vector fields can represent the same tangent vector, but, two fields are in the same equivalence class iff they differ by a field tangential to Γ_0.

The following notation will be used from now on

$$\langle d_{\Gamma_0} F, \nu_\varphi \rangle = [\varphi^* F],$$

for the tangential of the section F at Γ_0.

3.1 Examples

(a) As a first example, consider F to be a smooth section of the Banach space "bundle"[1]

$$E = \coprod_{\Gamma \in \mathcal{U}^{2+\alpha}} buc^{2+\alpha}(\Gamma),$$

[1] The term is used in a somewhat loose way here in order to appeal to intuition. A formal justification would require additional work that is not necessary for the purposes of this paper.

where F is smooth at Γ_0 if $[s \to F \circ \Gamma_s]$ is smooth for any smooth curve in $\mathcal{U}^{2+\alpha}$. As a specific example, take $f \in \mathrm{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and define

$$F(\Gamma) = f\big|_\Gamma.$$

Then F is a smooth section and

$$\langle d_{\Gamma_0} F, \nu_\varphi \rangle = \frac{d}{ds}\Big|_{s=0} f \circ \varphi_s\big|_{\Gamma_0} = \nabla f \cdot \nu_\varphi\big|_{\Gamma_0} = \partial_{\nu_\varphi} f\big|_{\Gamma_0}.$$

Remark 3.2 Notice that any Γ in the neighborhood $\mathcal{U}^{2+\alpha}$ of Γ_0 is diffeomorphic to it. Now, if one fixes a smooth "background" flow $\overline{\varphi}$ transversal to Γ_0 (such as φ^δ for small $\delta > 0$), the considerations of the previous section yield a function $\bar\rho_\Gamma$ such that $\varphi_\Gamma = \overline{\varphi} \circ (\mathrm{id}, \bar\rho_\Gamma)$ is a diffeomorphism between Γ_0 and Γ. It gives a local trivialization of E via

$$buc^{2+\alpha}(\Gamma_0) \times \mathcal{U}^{2+\alpha} \to E,\ (g, \Gamma) \mapsto g \circ \varphi_\Gamma.$$

This provides additional justification for the approach via flows described above.

(b) Let $f \in \mathrm{C}^\infty(\mathbb{R}^n)$ and consider

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{3.1}$$

and let

$$F : \mathcal{U}^{2+\alpha} \to \coprod_{\Gamma \in \mathcal{U}^{2+\alpha}} bvp^2(\Gamma),\ \Gamma \mapsto (-\Delta_\Omega, \gamma_\Gamma, f)$$

be the section of the "second order boundary value problems bundle" over $\mathcal{U}^{2+\alpha}$ corresponding to the above boundary value problem. Then

$$\langle d_{\Gamma_0} F, \nu_\varphi \rangle = \frac{d}{ds}\Big|_{s=0} \Big(\varphi_s^*(-\Delta_{\Omega_s})\varphi_*^s,\ \varphi_s^* \gamma_{\Gamma_s} \varphi_*^s,\ \varphi_s^* f \Big),$$

where $\varphi_*^s = (\varphi_s^*)^{-1} = (\varphi_s^{-1})^*$ and $\Omega_s = \varphi_s(\Omega_0)$.

Remark 3.3 Notice that, in this case, it is assumed that φ_s be defined everywhere so as to be able to transform the operator $-\Delta$ defined on Ω. Observe also that there is nothing geometric (that is, no identification/trivialization is necessary) in pulling the problem back to the domain Ω_0 since the original problem on Ω_s is equivalent to

$$\begin{cases} -\varphi_s^* \circ \Delta \circ \varphi_*^s(v) = \varphi_s^* f & \text{in } \Omega_0, \\ v = 0 & \text{on } \Gamma_0, \end{cases} \tag{3.2}$$

for $v = \varphi_s^* u$. This also provides justification for the use of the pull-back trivialization introduced earlier as it perfectly matches the definition of tangential by means of pull-backs. To be more explicit: if one is interested, as is the case here, in computing $\langle d_{\Gamma_0} u, \nu_\varphi \rangle$, then one needs to consider $\frac{d}{ds}\big|_{s=0} \varphi_s^* u$, which necessarily involves determining the solution $v = \varphi_s^* u$ of (3.2).

Remark 3.4 Observe that, since only $\nu_\varphi\big|_{\Gamma_0}$ matters, there arises great freedom in the choice of an extension of the vector field. This freedom leads to the intuition that, choosing the "trivial extension", the interior of the problem should not have an influence on the domain variation other than through u_0, the solution in Ω_0. More on this aspect later.

Returning to the example, one can describe how the solution u depends on Γ or, with moving boundary problems in mind, how $\partial_{\nu_\Gamma} u$ depends on it. Fix a flow φ_s and solve (3.1) on Ω_s or (3.2) to obtain $u = u(s)$ or $v = v(s) = \varphi_s^* u(s)$, respectively. Then

$$\frac{d}{ds}\Big|_{s=0} \varphi_s^* u = \frac{d}{ds}\Big|_{s=0} v(s),$$

as remarked above. Define $\mathcal{A}(s) = -\varphi_s^* \Delta \varphi_*^s$ so that

$$-\Delta\big(v \circ \varphi_s^{-1}\big)(x) = \big(\mathcal{A}(s)v\big)\big(\varphi_s^{-1}(x)\big),\ x \in \Omega_s,$$

for $v : \Omega_0 \to \mathbb{R}$, where

$$\Omega_0 \ni y = y_s(x) = \varphi_s^{-1}(x).$$

Using this notation it easily follows that

$$\mathcal{A}(s) = -\sum_{k,l=1}^n \underbrace{\left(\sum_{j=1}^n \frac{dy_s^l}{dx^j}\frac{dy_s^k}{dx^j}\right)}_{a_{kl}=} \frac{\partial^2}{\partial y^k \partial y^l} - \sum_{l=1}^n \underbrace{\left(\sum_{j=1}^n \frac{\partial^2 y_s^l}{(\partial x^j)^2}\right)}_{b_l=} \frac{\partial}{\partial y^l}.$$

Differentiating the equations yields

$$\frac{d}{ds}\Big|_{s=0} \mathcal{A}(s)u_0 + \mathcal{A}(0)\,\frac{d}{ds}\Big|_{s=0} u(s) = \frac{d}{ds}\big(f \circ \varphi_s\big) = \partial_{\nu_\varphi} f \text{ in } \Omega_0,$$

and that $\frac{d}{ds}\big|_{s=0} u(s) = 0$ on Γ_0. Next one needs an expression for $\frac{d}{ds}\big|_{s=0}$ of the coefficients $\frac{\partial \varphi_s^{-1}}{\partial x^j} \circ \varphi_s$ and $\frac{\partial^2 \varphi_s^{-1}}{(\partial x^j)^2} \circ \varphi_s$.

Lemma 3.5 *It holds that*

$$\frac{d}{ds}\Big|_{s=0} a_{kl} = -\big(D\nu_\varphi + D\nu_\varphi^\top\big)_{kl} \ \text{ and } \ b_l = -\Delta \nu_\varphi^l,\ k, l = 1, \dots, n$$

for the vector field ν_φ associated to the flow φ.

Proof It is plain that $\varphi_s \circ \varphi_s^{-1} = \mathrm{id}$ implies $D\varphi_s^{-1} \circ \varphi_s = (D\varphi_s)^{-1}$. Then

$$\begin{cases} \dot{\varphi}_s = \nu_\varphi(\varphi_s), \\ \varphi_0 = \mathrm{id}, \end{cases}$$

implies that

$$\begin{cases} D\dot{\varphi}_s = D\nu_\varphi(\varphi_s)D\varphi_s, \\ D\varphi_0 = \mathbb{1}. \end{cases}$$

It follows that

$$\frac{d}{ds}\Big|_{s=0} (D\varphi_s^{-1} \circ \varphi_s) = -(D\varphi_s)^{-1} D\dot{\varphi}_s (D\varphi_s)^{-1}\big|_{s=0} = -D\dot{\varphi}_0 = -D\nu_\varphi.$$

Next, the relation $D\varphi_s^{-1} = (D\varphi_s)^{-1}$ entails that

$$D(D\varphi_s^{-1}) = -(D\varphi_s)^{-1} D^2\varphi_s (D\varphi_s)^{-1}.$$

It also holds that

$$\begin{cases} D(D\varphi_s)^{\cdot} = (D^2\varphi_s)^{\cdot} = D^2\nu_\varphi \circ \varphi_s\big(D\varphi_s, D\varphi_s\big) + D\nu_\varphi \circ \varphi_s D^2\varphi_s, \\ D^2\varphi_0 = 0, \end{cases}$$

This yields

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0}\big(D(D\varphi_s^{-1})\circ\varphi_s\big) = -\frac{d}{ds}\Big|_{s=0} & (D\varphi_s)^{-1}D^2\varphi_0(D\varphi_0)^{-1} - (D\varphi_0)^{-1}D^2\dot\varphi_0(D\varphi_0)^{-1} + \\ & -\frac{d}{ds}\Big|_{s=0}(D\varphi_0)^{-1}D^2\varphi_0(D\varphi_s)^{-1} = -D^2\nu_\varphi(\mathbb{1},\mathbb{1}) = -D^2\nu_\varphi. \end{aligned}$$

The claim easily follows. □

Summarizing one has that

$$\begin{aligned} \mathcal{A}(0) &= -\Delta \text{ on } \Omega_0, \\ \frac{d}{ds}\Big|_{s=0}\varphi_s^* f &= \partial_{\nu_\varphi} f = \nabla f \cdot \nu_\varphi, \\ \frac{d}{ds}\Big|_{s=0}\mathcal{A}(0) &= \sum_{k,l=1}^n \big(D\nu_\varphi + D\nu_\varphi^\top\big)_{lk}\frac{\partial^2}{\partial y^k\partial y^l} + \sum_{l=1}^n \Delta\nu_\varphi^l\frac{\partial}{\partial y^l}. \end{aligned}$$

Finally it is arrived at

Theorem 3.6 *It holds that*

$$\frac{d}{ds}\Big|_{s=0}\varphi_s^*\big(u(s)\big) = (-\Delta_{\Omega_0},\gamma_{\Gamma_0})^{-1}\Big(\nu_\varphi\cdot\nabla f - 2D\nu_\varphi : D^2u_0 - \Delta\nu_\varphi\cdot\nabla u_0, 0\Big), \tag{3.3}$$

that is, the solution of the homogeneous Dirichlet problem with the given data.

In the above theorem the notation $A : B$ was used for $\mathrm{tr}(A^\top B)$ and symmetric matrices A, B. Next consider $\partial_{\nu_\Gamma}u$. As the normal derivative is a function on the boundary, it is natural to expect $\frac{d}{ds}\big|_{s=0}\partial_{\nu_{\Gamma_s}}u$ not to depend on interior (to the domain Ω_0) information other than u_0 itself. Using representation (3.3), however, would seem to indicate that there be dependence on $\dot\varphi_s\big|_{\Omega_0}$ as well. It is therefore best to proceed in a slightly different way. Take u_0, the solution of (3.1) in Ω_0, and assume, at first, that φ flows into Ω_0, and look for $u(s) = u_0 + \bar u(s)$. Then $\bar u$ satisfies

$$\begin{cases} -\Delta\bar u = 0 & \text{in } \Omega_s, \\ \bar u = -u_0\big|_{\Gamma_s} & \text{on } \Gamma_s \end{cases} \tag{3.4}$$

and, consequently one has that

$$\partial_{\nu_{\Gamma_s}}u(s) = \partial_{\nu_{\Gamma_s}}u_0 + \partial_{\nu_{\Gamma_s}}\bar u(s).$$

Next observe that $\bar u(0) \equiv 0$ and that

$$\partial_{\nu_{\Gamma_s}}\bar u(s) = -DtN_{\Gamma_s}\big(u_0\big|_{\Gamma_s}\big),$$

where DtN_Γ denotes the standard Dirichlet-to-Neumann operator of the domain Ω with boundary Γ. It can be concluded that

$$\frac{d}{ds}\Big|_{s=0}\varphi_s^*\big(\partial_{\nu_{\Gamma_s}}u(s)\big) = \Big(\cancelto{0}{\frac{d}{ds}\Big|_{s=0}\varphi_s^*\nu_{\Gamma_s}\Big)}\cdot\nabla u_0 + \partial_{\nu_0}\frac{d}{ds}\Big|_{s=0}\varphi_s^*\Big(u_0\big|_{\Gamma_s}\Big) +$$

$$
\begin{aligned}
&- \left(\frac{d}{ds}\Big|_{s=0} \varphi_s^* DtN_{\Gamma_s} \varphi_*^s \right) \left(\cancelto{0}{u_0\big|_{\Gamma_0}} \right) \\
&- DtN_{\Gamma_0} \left(\frac{d}{ds}\Big|_{s=0} \varphi_s^* \left(u_0\big|_{\Gamma_s} \right) \right) \\
&= \partial_{\nu_0} \partial_{\nu_\varphi} u_0 - DtN_{\Gamma_0} \left(\partial_{\nu_\varphi} u_0 \right).
\end{aligned}
$$

The first term vanishes in view of Lemma 4.3 below which shows that the domain variation of the normal is a combination of tangent vectors.

Theorem 3.7 *It holds that*

$$
\langle d_{\Gamma_0} \partial_{\nu_\Gamma} u, \nu_\varphi \rangle = \frac{d}{ds}\Big|_{s=0} \varphi_s^* \partial_{\nu_{\Gamma_s}} u(s) = \partial_{\nu_0} \partial_{\nu_\varphi} u_0 - DtN_{\Gamma_0} \big(\partial_{\nu_\varphi} u_0 \big). \tag{3.5}
$$

Proof It remains to show that the claim is valid for a general flow φ. First choose an outward flow $\bar{\varphi}$ and define

$$
\overline{\Omega}_\varepsilon = \bar{\varphi}_\varepsilon(\Omega_0), \ \varepsilon > 0.
$$

Then, given an arbitrary flow φ, it will hold that

$$
\varphi_s(\Omega_0) \subset \overline{\Omega}_\varepsilon \text{ if } s << 1.
$$

Denoting by $\bar{u}_\varepsilon$ the solution of (3.1) in $\overline{\Omega}_\varepsilon$, look for $u = \bar{u}_\varepsilon + \bar{w}$, so that $\bar{w}$ is harmonic in Ω_s and satisfies

$$
\bar{w} = -\bar{u}_\varepsilon \text{ on } \Gamma_s,
$$

at least for $s << 1$. Retracing the steps of the computation preceding the theorem, it is arrived at

$$
\begin{aligned}
\frac{d}{ds}\Big|_{s=0} \varphi_s^* \partial_{\nu_{\Gamma_s}} u(s) = \frac{d}{ds}\Big|_{s=0} & \varphi_s^* \left(\partial_{\nu_{\Gamma_s}} \bar{u}_\varepsilon \right) + \\
& - \left(\frac{d}{ds}\Big|_{s=0} \varphi_s^* DtN_{\Gamma_s} \varphi_*^s \right) \left(\bar{u}_\varepsilon\big|_{\Gamma_0} \right) - DtN_{\Gamma_0} \left(\frac{d}{ds}\Big|_{s=0} \bar{u}_\varepsilon\big|_{\Gamma_s} \right).
\end{aligned}
$$

As this last formula is valid for any $\varepsilon > 0$, it can be inferred that, letting $\varepsilon \to 0$, the claim is indeed valid since $\bar{u}_0 = u_0$ and thus $\bar{u}_0\big|_{\Gamma_0} \equiv 0$. □

This shows that, if $\frac{d}{ds}\big|_{s=0} \varphi_s^*(\partial_{\nu_{\Gamma_s}} u)$ is computed by means of Theorem 3.3, then its independence of $\dot{\varphi}_s\big|_{\Omega_0}$ is obfuscated. There indeed would even appear a possible dependence on $D^2 \nu_\varphi$. This can make calculations for moving boundary problems less transparent and more cumbersome.

4 Variations in a parametrized context

For a given smooth flow φ one always has that

$$
\varphi_s(\Gamma_0) \subset \mathcal{T}_{r_0}^{\nu^\delta}(\Gamma_0)
$$

for $s << 1$. Then Lemma 2.6 implies that

$$
\varphi_s(\Gamma_0) = \left\{ \varphi^\delta \big(y, \rho(s, y) \big) \,\big|\, y \in \Gamma_0 \right\} = \varphi^\delta \circ \big(\mathrm{id}, \rho(s, \cdot) \big)(\Gamma_0) =: \Gamma_{\rho(s,\cdot)},
$$

for some $\rho(s,\cdot)\in buc^{2+\alpha}(\Gamma_0)$. It follows that, in calculations, φ_s can be replaced by

$$\varphi_\rho := \Big[s\mapsto \varphi^\delta\big(\cdot,\rho(s,\cdot)\big)\Big],$$

which is a family of diffeomorphisms tracing the same curve of hypersurfaces. Notice that

$$d\varphi_\rho = d\varphi^\delta(\cdot,\rho)+\dot{\varphi}^\delta d\rho$$

clearly shows that these are, indeed, diffeomorphisms, provided ρ is small enough in the C^1-topology. These "flows" differ merely in their (irrelevant) tangential action. It holds that

$$\frac{d}{ds}\Big|_{s=0}\varphi^\delta\big(\cdot,\rho(s,\cdot)\big)=\dot{\varphi}^\delta\dot{\rho}(0,\cdot)=:\dot{\rho}_0\,\nu^\delta$$

on Γ_0. If, on occasion, an extension to a flow on $\mathbb{R}^n$, denoted by Φ_ρ, is needed, one can choose one of infinitely many extensions. Here, for the sake of definiteness, it is proceeded as follows: for $x\in\mathcal{T}_{r_0}^{\nu^\delta}(\Gamma_0)^{\mathsf{c}}$ simply set

$$\Phi_{\rho,s}(x)=\Phi_\rho(s,x)\equiv x,\ s\in(-\varepsilon,\varepsilon),$$

while in $\mathcal{T}_{r_0}^{\nu^\delta}(\Gamma_0)$, using the coordinates $x=\big(y(x),r(x)\big)$ given by Lemma 2.5, define

$$\Phi_{\rho,s}(y,r)=\Phi_\rho\big((y,r),s\big)=\big(y,r+\rho(s,y)\eta(r)\big), \tag{4.1}$$

where η is a cut-off function of the type used in the proof of Lemma 2.5. Clearly $\Phi_{\rho,\cdot}$ is a family of diffeomorphisms, which is as smooth as ρ is, and

$$\Phi_{\rho,s}\big|_{\Gamma_0}=\varphi^\delta\circ\big(\mathrm{id},\rho(s,\cdot)\big).$$

Again this rests on the assumption that ρ is small which makes the map $[r\mapsto r+\rho(s,y)\eta(r)]$ invertible for fixed (s,y) thanks to its monotonicity.

Remark 4.1 In calculations it is often convenient to replace $\rho(s,\cdot)$ by $s\dot{\rho}_0$ in the above definition as one obtains a curve of hypersurfaces that is, yes, different, but generates the same tangent vector.

The above considerations can be be summarized as follows.

Proposition 4.2 *It holds that*

$$T_{\Gamma_0}\mathcal{M}^{2+\alpha}\mathrel{\hat{=}}buc^{2+\alpha}(\Gamma_0),$$

where the superscript over the equal sign indicates an identification, which, in this case, is via the map

$$h\mapsto h\,\dot{\varphi}^\delta=h\,\nu^\delta,\ buc^{2+\alpha}(\Gamma_0)\to T_{\Gamma_0}\mathcal{M}^{2+\alpha}$$

Proof Notice that

$$\frac{d}{ds}\Big|_{s=0}\varphi_s\big|_{\Gamma_0}=\nu_\varphi\big|_{\Gamma_0},$$

as well as that

$$\frac{d}{ds}\Big|_{s=0}\varphi^\delta\circ\big(\mathrm{id},\rho(s,\cdot)\big)=\nu^\delta\big|_{\Gamma_0}\dot{\rho}(0,\cdot).$$

Now, since φ_s and $\varphi^\delta \circ \big(\mathrm{id}, \rho(s,\cdot)\big)$ yield the same curve of hypersurfaces, the vector fields $\nu_\varphi\big|_{\Gamma_0}$ and $\dot\rho_0 \nu^\delta\big|_{\Gamma_0}$ represent the same tangent vector and, since φ can be any flow, the whole tangent space can be generated in this way. Furthermore the fields ν_φ and $\nu_{\tilde\varphi}$ associated with two smooth flows φ and $\tilde\varphi$ generating two distinct tangent vectors necessarily differ in their normal components at some point of Γ_0. In this case, their components in direction of the everywhere transversal field ν^δ will be different, too, showing that the map is injective. □

4.1 Variations of the normal vector

Denote the unit outward normal to Γ_ρ by ν_ρ for any given $\rho \in buc^{2+\alpha}(\Gamma_0)$. The preceding considerations and examples point to the necessity of computing $\langle d_{\Gamma_0}\nu_\Gamma, h\nu^\delta\rangle$. According to the above observations, this can be performed by evaluating $\frac{d}{ds}\big|_{s=0} \varphi^*_{sh}\nu_{sh}$ for $h \in buc^{2+\alpha}(\Gamma_0)$

Lemma 4.3 *It holds that*

$$\frac{d}{ds}\Big|_{s=0} \varphi^*_{sh}\,\nu_{sh} = h\sum_{j=1}^{n-1}\big(\langle d_y\nu^\delta(y), \tau^0_j\rangle\cdot\nu_0\big)\,\tau^0_j - \big(\nu^\delta|\nu_0\big)\sum_{j=1}^{n-1}\partial_j h\,\tau^0_j,$$

that is, a differential operator of order 1 acting on h. Recall that, by construction, $\nu^0\big|_{\Gamma_0} = \nu_0 = \nu_{\Gamma_0}$.

Proof The notation $\tilde\tau^\rho_j$, $j = 1,\dots,n-1$ introduced in (2.4) is used here for a basis of tangent vectors in $T\Gamma_\rho$ and τ^ρ_j, $j = 1,\dots,n-1$ for their normalized counterparts. It then follows from

$$|\nu_{sh}| = 1 \quad\text{and}\quad \tau^{sh}_j\cdot\nu_{sh} = 0 \quad\text{for}\quad j = 1,\dots,n-1.$$

that

$$\begin{aligned}\Big(\frac{d}{ds}\nu_{sh}\Big)\cdot\tau^{sh}_j &= -\Big(\frac{d}{ds}\tau^{sh}_j\Big)\cdot\nu_{sh}\\ \Big(\frac{d}{ds}\nu_{sh}\Big)\cdot\nu_{sh} &= 0.\end{aligned}$$

Evaluating in $s = 0$ yields

$$\frac{d}{ds}\Big|_{s=0}\nu_{sh} = -\sum_{j=1}^{n-1}\Big(\frac{d}{ds}\Big|_{s=0}\tau^{sh}_j\cdot\nu_0\Big)\,\tau^0_j,$$

where, by design, τ^0_j, $j = 1,\dots,n-1$ is a basis of $T\Gamma_0$. Thus it is enough to compute $\frac{d}{ds}\big|_{s=0}\tau^{sh}_j$ for $j = 1,\dots,n-1$ in order to compute $\frac{d}{ds}\big|_{s=0}\nu_{sh}$. Next observe that

$$\frac{d}{ds}\Big|_{s=0}\widetilde\tau^{sh}_j = \frac{d}{ds}\Big|_{s=0}\Big(|\widetilde\tau^{sh}_j|\,\tau^{sh}_j\Big) = \frac{1}{|\widetilde\tau^0_j|}\Big(\frac{d}{ds}\Big|_{s=0}\widetilde\tau^{sh}_j\cdot\widetilde\tau^0_j\Big)\,\tau^0_j + |\widetilde\tau^0_j|\,\frac{d}{ds}\Big|_{s=0}\tau^{sh}_j,$$

where $\widetilde\tau^0_j = \tau^0_j$ is a unit vector for $j = 1,\dots,n-1$. Consequently one has that

$$\frac{d}{ds}\Big|_{s=0}\tau^{sh}_j = \frac{d}{ds}\Big|_{s=0}\widetilde\tau^{sh}_j - \Big(\frac{d}{ds}\Big|_{s=0}\widetilde\tau^{sh}_j\cdot\tau^0_j\Big)\,\tau^0_j,\ j = 1,\dots,n-1,$$

which shows that it is, in fact, enough to compute $\frac{d}{ds}\big|_{s=0}\widetilde{\tau}_j^{sh}$ for $j=1,\dots,n-1$. Now

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0}\widetilde{\tau}_j^{sh}(y,sh(y)) &= \frac{d}{ds}\bigg|_{s=0}\langle d_y\varphi^\delta(y,sh(y)),\tau_j^0\rangle \\ &\quad+\frac{d}{ds}\bigg|_{s=0}\left[\nu^\delta\circ\varphi^\delta(y,sh(y))\,s\partial_j h(y)\right] \\ &= \langle d_y\nu^\delta(y),\tau_j^0\rangle\,h(y)+\nu^\delta(y)\partial_j h(y),\ \ y\in\Gamma_0,\end{aligned}$$

for $j=1,\dots,n-1$, and then

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0}\tau_j^{sh}\big(y,sh(y)\big) &= \Big[\big\langle d_y\nu^\delta(y),\tau_j^0\big\rangle-\Big(\big\langle d_y\nu^\delta(y),\tau_j^0\big\rangle\cdot\tau_j^0\Big)\,\tau_0^j\Big]\,h(y) \\ &\quad+\Big[\nu^\delta(y)-\big(\nu^\delta(y)\cdot\tau_j^0\big)\tau_j^0\Big]\,\partial_j h(y),\quad y\in\Gamma_0.\end{aligned}$$

Finally

$$\frac{d}{ds}\bigg|_{s=0}\tau_j^{sh}\cdot\nu_0=\Big(\big\langle d_y\nu^\delta(y),\tau_j^0\big\rangle\cdot\nu_0\Big)\,h+(\nu^\delta\cdot\nu_0)\,\partial_j h$$

and thus

$$\frac{d}{ds}\bigg|_{s=0}\nu_{sh}=\left(\sum_{j=1}^{n-1}\big\langle d_y\nu^\delta(y),\tau_j^0\big\rangle\cdot\nu_0\right)h\,\tau_j^0-(\nu^\delta\cdot\nu_0)\sum_{j=1}^{n-1}(\partial_j h)\,\tau_j^0$$

as claimed. □

Remark 4.4 Notice that $\langle d_y\nu^\delta(y),\tau_j^0\rangle\simeq\lambda_j^0\tau_j^0$ for $\delta\simeq 0$ since $\nu^\delta\simeq\nu_0$. Recall that λ_j are the principal curvatures of Γ_0.

Remark 4.5 It is interesting to observe that, while the smoothness of the reference flow φ^δ was necessary for the calculations of Sect. 3, it can be replaced here by the flow $\widetilde{\varphi}$ generated by $\widetilde{\nu}=\nabla\big(d_{\Gamma_0}(\psi\circ d_{\Gamma_0})\big)\in buc^{1+\alpha}$, where the smoothness follows from Remark 2.4 and ψ is a smooth cut-off function with

$$\psi\big|_{[-r_0/2,r_0/2]}\equiv 1\quad\text{and}\quad\psi\big|_{(-3r_0/4,3r_0/4)^c}\equiv 0.$$

Performing the same calculations as in the proof of Lemma 4.3 and using the same notation, it is seen that

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0}\varphi_{sh}^*\nu_{sh} &= h\sum_{j=1}^{n-1}\Big(\big\langle d_y\nu_0(y),\tau_j^0\big\rangle\big|\nu_0\Big)\,\tau_j^0-\big(\nu_0|\nu_0\big)\sum_{j=1}^{n-1}\partial_j h\,\tau_j^0 \\ &= \sum_{j=1}^{n-1}\partial_j h\,\tau_j^0,\end{aligned}\tag{4.2}$$

since $\widetilde{\nu}\big|_{\Gamma_0}=\nu_0$ and $\langle d_y\nu_0(y),\tau_j^0\rangle=\lambda_j^0\tau_j^0$.

If $\rho(t,\cdot)$ is a time dependent function, then one can compute the velocity V in normal direction of the corresponding domains $\Gamma_{\rho(t,\cdot)}$. This is clearly an important quantity for moving boundary problems. One has

$$\begin{aligned} V(y) &= \frac{d}{dt}\varphi^\delta\big(y,\rho(\cdot,y)\big)\cdot\nu_{\rho(\cdot,y)} \\ &= \left[\frac{d}{dr}\varphi^\delta\big(y,\rho(\cdot,y)\big)\cdot\nu_{\rho(\cdot,y)}\right]\rho_t(\cdot,y) \\ &= \big[(\nu^\delta\circ\varphi^\delta)\big(y,\rho(\cdot,y)\big)\cdot\nu_{\rho(\cdot,y)}\big]\ \rho_t(\cdot,y), \\ &\qquad\qquad \text{for } y\in\Gamma_0,\quad \text{i.e. } \varphi^\delta\big(y,\rho(\cdot,y)\big)\in\Gamma_\rho, \end{aligned}$$

or, for short, $V=(\nu^\delta\cdot\nu_\rho)\,\rho_t$. Notice that

$$\big(\nu^\delta\circ\varphi^\delta\cdot\nu_\rho\big)\big(y,\rho(t,y)\big)\simeq\nu_{\Gamma_0}\cdot\nu_{\Gamma_0}=1,$$

uniformly in $y\in\Gamma_0$, if $\delta<<1$ and $t\simeq 0$. If using the alternative, less regular flow $\widetilde{\varphi}$, the same calculation yields $V=\rho_t$.

Lemma 4.6 *It holds that*

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0}\nu^\delta\circ\varphi^\delta\circ(\mathrm{id},sh)\cdot\nu_{sh} &= \langle d\nu^\delta,\nu^\delta\rangle\cdot\nu_0\,h \\ &\quad+\sum_{j=1}^{n-1}\big(\nu^\delta\cdot\tau_j^0\big)\big(\big\langle d_y\nu^\delta(y),\tau_j^0\big\rangle\cdot\nu_0\big)\,h \\ &\quad-\sum_{j=1}^{n-1}(\nu^\delta\cdot\nu_0)\big(\nu^\delta\cdot\tau_j^0\big)\,\partial_j h. \end{aligned}$$

Proof A direct computation using Lemma 4.3 gives that the desired variation amounts to

$$\left\langle d\nu^\delta,\frac{d}{ds}\Big|_{s=0}\varphi^\delta\right\rangle\cdot\nu_0+\nu^\delta\cdot\frac{d}{ds}\Big|_{s=0}\nu_{sh}=\big[\langle d\nu^\delta,\nu^\delta\rangle\cdot\nu_0\big]h-\sum_{j=1}^{n-1}(\nu^\delta\cdot\nu_0)\left(\nu^\delta\cdot\tau_j^0\right)\partial_j h$$

as stated. □

Remark 4.7 Notice that, when $\delta<<1$, one has that

$$\langle d\nu^\delta,\nu^\delta\rangle\simeq 0,\ \ \nu^\delta\cdot\tau_j^0\simeq 0, \text{ and } \nu^\delta\cdot\nu_0\simeq 1,$$

uniformly on Γ_0. It should also be pointed out that this variation vanishes if δ can be set to zero. This means that, for $\widetilde{\nu}$ replacing ν^δ, one has that

$$\frac{d}{ds}\Big|_{s=0}\widetilde{\nu}\circ\widetilde{\varphi}\circ(\mathrm{id},sh)\cdot\nu_{sh}=0,$$

using Remark 4.5 instead of Lemma 4.3.

4.2 Examples revisited

It is of course possible to interpret the variation of the solution of a boundary value problem as in Example (b) of Sect. 3.1 in terms of the identification of Proposition 4.2.

Corollary 4.8 *Given a smooth flow φ, let φ_{sh} be the corresponding equivalent curve of diffeomorphisms introduced just before Proposition 4.2. Denote the extension described in (4.1) by $\Phi=\Phi_{sh}$ and by $\nu_{sh}=\nu_\Phi$ the corresponding vector field, for which one has that*

$$\nu_\Phi=(h\circ y)(\eta\circ r)\nu^\delta\circ\varphi^\delta,$$

in the tubular neighborhood where it is not trivial. Here y, r, and φ^δ are to be thought of as functions of the variable x. Notice that ν_Φ is an extension of ν_φ. Then it is already known that ν_φ in (3.3) can be replaced by $(h \circ y)(\eta \circ r)\, \nu^\delta$, which, incidentally, coincides with $h\nu^\delta$ on Γ_0. The additional terms $D\nu_\varphi$ and $D^2\nu_\varphi$ can be replaced by

$$D\left(h\eta\, \nu_0^\delta\right)(x) = (h \circ y)(x)(\eta \circ r)(x) D\nu^\delta(x) + D\big(h \circ y\, \eta \circ r\big)(x)\nu^\delta(x)$$

and

$$\begin{aligned} D^2(h\eta\, \nu^\delta)(x) = (h \circ y)(x)(\eta \circ r)(x) D^2\nu^\delta(x) + 2D\nu^\delta(x) D(h \circ y\, \eta \circ r)(x) \\ + D^2(h \circ y\, \eta \circ r)(x)\nu^\delta(x), \end{aligned}$$

respectively. Notice that, since $h : \Gamma_0 \to \mathbb{R}$ depends on y only, all of its non vanishing derivatives are tangential ones.

Remark 4.9 The corollary shows how convenient it is to think in terms of flows or curves of diffeomorphisms: calculations can be performed in $\mathbb{R}^n$ and not on the surface. Eventually one can replace the generic flow with a parametrized one by means of Proposition 4.2 and the coordinates of Lemma 2.5 to obtain concrete expressions in terms of the parameter function ρ. Recall that $h = \dot\rho(0, \cdot)$.

Remark 4.10 It should be pointed out that, when the surface Γ_0 is smooth, then δ can be chosen to vanish (no regularization needed). In that case $\nu^0\big|_{\Gamma_0} = \nu_0$, and consequently, the terms $D\nu^0$ and $D^2\nu^0$ have geometric interpretations. E.g. $D\nu^0$ contains information about the curvatures of Γ_0 and its Christoffel symbols.

5 Moving boundary problems

A well-known classical moving boundary problem is the Hele-Shaw problems. It is used in this section as a prototypical example to illustrate the benefits of the linearization approach described the preceding sections which include conciseness and transparency.

5.1 Hele-Shaw type problem

Consider the system

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega(t) \quad \text{for} \quad t > 0, \\ u = 0 & \text{on } \Gamma(t) \quad \text{for} \quad t > 0, \\ V = -\partial_\nu u & \text{on } \Gamma(t) \quad \text{for} \quad t > 0, \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{5.1}$$

for $\Gamma_0 \in buc^{2+\alpha}$. Then one has the following

Proposition 5.1 *The linearization of (5.1) in $(u(t), \Gamma(t)) \equiv (u_0, \Gamma_0)$, where clearly u_0 is the solution of the Poisson equation with right-hand-side 1 and homogeneous Dirichlet condition in Ω_0, is given by*

$$\begin{cases} -\Delta \bar w = 0 & \text{in } \Omega_0, \quad \text{for} \quad t > 0 \\ \bar w = -\partial_{\nu_\varphi} u_0 & \text{on } \Gamma_0 \quad \text{for} \quad t > 0, \\ \dot\nu_\varphi \cdot \nu_0 = -\partial_{\nu_0}\partial_{\nu_\varphi} u_0 - \partial_{\nu_0}\bar w & \text{on } \Gamma_0 \quad \text{for} \quad t > 0, \\ \nu_\varphi(0) = 0, \end{cases} \tag{5.2}$$

where ν_φ denotes the time dependent variation vector field used to infinitesimaly deform Γ_0 and $\dot\nu_\varphi$ its time derivative (see proof below for more detail). In particular, if

$$\nu_\varphi = h\,\nu^\delta, \text{ then } \dot\nu_\varphi = \dot h\,\nu^\delta,$$

and (5.2) *reduces to*

$$\begin{cases} -\Delta\bar w = 0 & \text{in } \Omega_0 \ \text{ for } \ t>0,\\ \bar w = -(\partial_{\nu^\delta}u_0)h & \text{on } \Gamma_0 \ \text{ for } \ t>0,\\ (\nu^\delta\cdot\nu_0)\dot h = -(\partial_{\nu_0}\partial_{\nu^\delta}u_0)h - \partial_{\nu_0}\bar w & \text{on } \Gamma_0 \ \text{ for } \ t>0,\\ h(0,\cdot) = 0 & \text{on } \Gamma_0.\end{cases}$$

Remark 5.2 Since the right hand side of Poisson's equation in (5.1) is positive, the strong maximum principle implies that

$$\partial_{\nu_0}u_0 < 0,$$

and consequently the same inequality holds for $\partial_{\nu^\delta}u_0$ since δ can be chosen arbitrarily small. This can be used to show that the operator

$$h \mapsto DtN_{\Gamma_0}\big((\partial_{\nu^\delta}u_0)h\big),\ buc^{2+\alpha}(\Gamma_0)\to buc^{1+\alpha}(\Gamma_0)$$

generates an analytic semigroup as required by maximal regularity theory to obtain a solution of the corresponding nonlinear problem. In this case the linearized system reduces to the single equation

$$(\nu^\delta\cdot\nu_0)\,\dot h = DtN_{\Gamma_0}\big((\partial_{\nu^\delta}u_0)h\big) - (\partial_{\nu_0}\partial_{\nu^\delta}u_0)h. \tag{5.3}$$

Remark 5.3 One has that

$$\partial_{\nu^\delta}u_0 \in buc^{2+\alpha}.$$

This regularity is needed to ensure that multiplication with this normal derivative of u_0 is a continuous operation on $buc^{2+\alpha}(\Gamma_0)$ and then obtain the generation result of the previous remark.

Proof Observe that

$$\begin{aligned}\Delta\big(\partial_{\nu^\delta}u_0\big) = \Delta\big(\nu^\delta\cdot\nabla u_0\big) &= \sum_{j=1}^n \partial_j^2\big(\nu_k^\delta\partial_k u_0\big)\\ &= \sum_{j=1}^n\big(\partial_j^2\nu_k^\delta\partial_k u_0 + 2\partial_j\nu_k^\delta\partial_j\partial_k u_0 + \nu_k^\delta\partial_k\underbrace{\Delta u_0}_{=-1}\big) \in buc^\alpha(\Omega_0),\end{aligned}$$

and that

$$\begin{aligned}\partial_{\nu_0}\partial_{\nu^\delta}u_0 &= \nu_0^k\partial_k\left(\left[(\nu^\delta\cdot\nu_0)\nu_0 + \sum_{j=1}^{n-1}\big(\nu^\delta\cdot\tau_j^0\big)\tau_j^0\right]\cdot\nabla u_0\right)\\ &= \partial_{\nu_0}(\nu^\delta\cdot\nu_0)\partial_{\nu_0}u_0 + (\nu^\delta\cdot\nu_0)\partial_{\nu_0\nu_0}u_0\\ &= (\partial_{\nu_0}\nu^\delta\cdot\nu_0)\partial_{\nu_0}u_0 - (\nu^\delta\cdot\nu_0) \in buc^{1+\alpha}(\Gamma_0),\end{aligned}$$

since

$$\partial_{\nu_0}\nu_0 = 0 \quad \text{and} \quad \partial_{\nu_0\nu_0}u_0 + \cancelto{0}{\sum_{j=1}^{n-1}\partial_{\tau_j^0\tau_j^0}u_0} = -1$$

□

Remark 5.4 The proof of Remark 5.3 shows, in particular, that $\partial_{\nu_0}u_0 \in buc^{2+\alpha}(\Gamma_0)$ and, for this specific system, the linearization simplifies further to

$$\dot h = DtN_{\Gamma_0}\big((\partial_{\nu_0}u_0)h\big) - (\partial_{\nu_0\nu_0}u_0)h.$$

Proof (of Proposition 5.1) Take a two parameter family of diffeomorphisms $\varphi_{s,t}$ such that

$$\varphi_{0,t}\big|_{\Gamma_0} \equiv \mathrm{id}_{\Gamma_0}$$

and set

$$\nu_\varphi = \frac{d}{ds}\bigg|_{s=0}\varphi_{s,t}\big|_{\Gamma_0} \text{ as well as } \dot\nu_\varphi = \frac{d}{dt}\frac{d}{ds}\bigg|_{s=0}\varphi_{s,t}\big|_{\Gamma_0}.$$

As follows from the proof of Proposition 3.7, it is possible to assume without loss of generality that the diffeomorphisms “flow” into Ω_0. Then rewrite (5.1) as

$$\begin{cases} -\Delta\bar u = 0 & \text{in } \Omega_{s,t} \quad \text{for} \quad t>0,\\ \bar u = -u_0\big|_{\Gamma_{s,t}} & \text{on } \Gamma_{s,t} \quad \text{for} \quad t>0,\\ V = -\partial_{\nu_{\Gamma_{s,t}}}(u_0+\bar u) & \text{on } \Gamma_{s,t} \quad \text{for} \quad t>0,\\ \Gamma(0) = \Gamma_0, \end{cases}$$

for $u = u_0 + \bar u$, where, again, u_0 is the solution of Poisson equation on Ω_0 with homogeneous Dirichlet condition on the boundary and

$$\Omega_{s,t} = \varphi_{s,t}(\Omega_0) \quad \text{and} \quad \Gamma_{s,t} = \varphi_{s,t}(\Gamma_0).$$

Then

$$\begin{cases} -\mathcal{A}(s)\bar v = -\varphi_{s,t}^*\Delta\varphi_*^{s,t}\bar v = 0 & \text{in } \Omega_0 \quad \text{for} \quad t>0,\\ \bar v = -\varphi_{s,t}^*\big(u_0\big|_{\Gamma_{s,t}}\big) & \text{on } \Gamma_0 \quad \text{for} \quad t>0, \end{cases}$$

for $\bar v = \varphi_{s,t}^*\bar u$ and

$$V = \frac{d}{dt}\varphi_{s,t}\cdot\varphi_{s,t}^*\nu_{\Gamma_{s,t}} = -\varphi_{s,t}^*\Big[\partial_{\nu_{\Gamma_{s,t}}}u_0\big|_{\Gamma_{s,t}} + \partial_{\nu_{\Gamma_{s,t}}}\bar u\Big] \text{ on } \Gamma_0 \quad \text{for} \quad t>0.$$

Taking a variation in s and evaluating in $s=0$ yields

$$\begin{cases} -\mathcal{A}(0)\bar w - \frac{d}{ds}\big|_{s=0}\mathcal{A}\cancelto{0}{\bar v(0)} = 0 & \text{in } \Omega_0 \quad \text{for} \quad t>0,\\ \bar w = -\partial_{\nu_\varphi}u_0 & \text{on } \Gamma_0 \quad \text{for} \quad t>0, \end{cases}$$

and

$$\dot\nu_\varphi\cdot\nu_0 + 0\cdot\frac{d}{ds}\bigg|_{s=0}\varphi_{s,t}^*\nu_{\Gamma_{s,t}} = -\partial_{\nu_0}\partial_{\nu_\varphi}u_0 - \partial_{\nu_0}\bar w \text{ on } \Gamma_0 \quad \text{for} \quad t>0,$$

for $\bar{w} = \frac{d}{ds}\big|_{s=0} \bar{v}$ since $\frac{d}{dt}\varphi_{0,t} \equiv 0$. This system reduces to the claimed one at the end of the proposition if

$$\nu_\varphi = h(t)\, \nu^\delta \quad \text{and} \quad \dot{\nu}_\varphi = \dot{h}(t)\, \nu^\delta.$$

Just use Lemma 2.5 to replace the generic curve of diffeomorphisms with the equivalent $\Phi_{\rho,s,t}$, introduced in (4.1) based on

$$\Phi_{\rho,s,t}\big|_{\Gamma_0} = \varphi^\delta \circ \big(\mathrm{id}, \rho(s,t,\cdot)\big),$$

satisfying

$$\Omega_{s,t} = \Phi_{\rho,s,t}(\Omega_0) \quad \text{and} \quad \Gamma_{s,t} = \Phi_{\rho,s,t}(\Gamma_0),$$

and such that $\frac{d}{ds}\big|_{s=0} \rho(0,t,\cdot) \equiv h(t,\cdot)$ and $\frac{d}{ds}\big|_{s=0} \dot{\rho}(0,t,\cdot) \equiv \dot{h}(t,\cdot)$. □

While using known results for nonlinear evolution equations [8,17], one readily obtains classical local well-posedness results for (5.1), an alternative approach is proposed here which simplifies the application of maximal regularity avoiding the use of a fixed-point argument.

Remark 5.5 It is worth observing that the linearization of the nonlocal, nonlinear evolution equation

$$\begin{cases} V_{\Gamma(t)} = -\partial_{\nu_{\Gamma(t)}} u_{\Gamma(t)}, & t > 0, \\ \Gamma(0) = \Gamma_0, \end{cases}$$

for the unknown $\Gamma(\cdot)$, derived from (5.1) by solving Poisson's equation to obtain $u_{\Gamma(t)}$, can alternatively be computed by means of the procedure outlined in Example 3.1(b). Introducing $v = \Phi^*_{sh} u$, (5.1) is equivalent to

$$\begin{cases} -\Phi^*_{sh} \Delta \Phi^{sh}_* v = 1 & \text{in } \Omega_0 \quad \text{for} \quad t > 0, \\ v = 0 & \text{on } \Gamma_0 \quad \text{for} \quad t > 0, \\ V_{sh} = -\partial_{\nu_{sh}} \Phi^{sh}_* v & \text{on } \Gamma_0 \quad \text{for} \quad t > 0, \\ h(0,\cdot) \equiv 0 & \text{on } \Gamma_0, \end{cases}$$

where, as before, Φ^*_{sh} denotes the pull-back by Φ_{sh} and Φ^{sh}_* its inverse. Taking the variation leads to

$$\begin{cases} -\Delta w = \frac{d}{ds}\big|_{s=0} \Phi^*_{sh} \Delta \Phi^{sh}_* u_0 =: \mathcal{A}'_0 h & \text{in } \Omega_0 \quad \text{for} \quad t > 0, \\ w = 0 & \text{on } \Gamma_0 \quad \text{for} \quad t > 0, \end{cases}$$

for $w = \frac{d}{ds}\big|_{s=0} v_{sh}$ and then to

$$(\nu^\delta \cdot \nu_0)\dot{h} = -\big(\nu_0 \big| \mathcal{B}'_0 h\big) - \partial_{\nu_0}(-\Delta)^{-1}\mathcal{A}'_0 h,$$

where similarly $\mathcal{B}'_0 h = \frac{d}{ds}\big|_{s=0} \Phi^*_{sh} \nabla \Phi^{sh}_* u_0$ and Δ^{-1} denotes the inverse of the Dirichlet Laplacian. Comparing this to (5.3) it can be concluded that

$$-\big(\nu_0 \big| \mathcal{B}'_0 h\big) + \partial_{\nu_0} \Delta^{-1} \mathcal{A}'_0 h = DtN_{\Gamma_0}\big((\partial_{\nu^\delta} u_0) h\big) - (\partial_{\nu_0} \partial_{\nu^\delta} u_0) h \tag{5.4}$$

5.2 About optimal regularity

This section serves the purpose of illustrating the use of maximal regularity results in order to prove the local well-posedness of nonlinear parabolic problems such as the above nonlocal nonlinear surface evolution associated to the Hele-Shaw problem. While the results are known, the approach taken here is different, more direct, and does not rely on the use of standard fixed-point arguments.

There are two main types of maximal regularity: continuous and L^p maximal regularity. While the first class contains the case of (possibly singular) Hölder continuity, only the purely continuous case is mentioned here. Continuous maximal regularity on $[0, T]$ is said to hold for an operator $A : E_1 \operatorname{dom}(A) \subset E_0 \to E_0$ on a Banach space E_0 with norm $\|\cdot\|_0$ iff the abstract Cauchy problem

$$\dot{u} = Au + f \text{ in } E_0 \quad \text{for} \quad t > 0,\ u(t) = u_0, \tag{5.5}$$

has a unique solution $u \in \mathbb{E}_1 := \mathrm{C}^1\big([0, T], E_0\big) \cap \mathrm{C}\big([0, T], E_1\big)$ satisfying

$$\|u\|_{\mathrm{L}^\infty([0,T],E_1)} + \|\dot{u}\|_{\mathrm{L}^\infty([0,T],E_0)} \leq c\big(\|u\|_1 + \|f\|_{\mathrm{L}^\infty([0,T]E_0)}\big)$$

for any $(u_0, f) \in E_1 \times \mathbb{E}_0 := E_1 \times \mathrm{C}\big([0, T], E_0\big)$, where $\|\cdot\|_1 = \|\cdot\|_0 + \|A\cdot\|_0$ is the graph norm on E_1. Maximal regularity of continuous type imposes severe restrictions on the space E_0. The space E_0 can not, in particular, be reflexive (see [2]). It is, however, known that continuous maximal regularity holds in little Hölder spaces (via [8]). Since these spaces of classical regularity are also well suited to formulate geometric regularity and Hölder functions can be restricted to boundaries without loss of regularity, their choice seemed natural for the context of this paper. Without going into unnecessary details, L^p maximal regularity is obtained replacing the space $\mathbb{E}_0$ and $\mathbb{E}_1$ by $\mathrm{L}^p\big([0, T], E_0\big)$ and $\mathrm{H}^{1,p}\big([0, T], E_0\big) \cap \mathrm{L}^p\big([0, T], E_1\big)$, respectively. The space of initial data E_1 needs also to be replaced by an appropriate interpolation space E_p. Some advantages of L^p maximal regularity are that it holds for a larger class of "base" spaces E_0, containing common Banach space scales for boundary value problems such as Sobolev–Slobodeckii spaces, and that it allows for somewhat less smooth initial data. It also introduces additional technicalities due to the loss of regularity incurred when taking traces. For free boundary problems, the spaces have to work well in combination with domain regularity as well which imposes restrictions on p, making the allowed functions quite regular and thus not far from the classical regularity context chosen here.

We now turn to the basic existence result mentioned above and remind the reader that it is typically proven by the use of fixed-point arguments.

Theorem 5.6 *Let $F \in \mathrm{C}^1(E_1, E_0)$ be such that $DF(u_0) \in \mathcal{L}(E_1, E_0)$ possesses continuous maximal regularity on $[0, T]$ for $T > 0$ and let $u_0 \in E_1$. Then the nonlinear Cauchy problem*

$$\dot{u} = F(u) \text{ in } E_0 \quad \text{for} \quad t > 0,\ u(0) = u_0, \tag{5.6}$$

has a unique local solution $u \in \mathrm{C}^1\big([0, T_0], E_0\big) \cap \mathrm{C}\big([0, T_0], E_1\big)$ for some $T_0 > 0$.

Proof By the open mapping theorem, maximal regularity is equivalent to the invertibility of the operator

$$\big(\partial_t - DF(u_0), \gamma_0\big) : \mathbb{E}_1 \to \mathbb{E}_0 \times E_1,$$

in the notation introduced above. Equivalently, it amounts to the invertibility of

$$\partial_t - DF(u_0) : \mathbb{E}_1^0 \to \mathbb{E}_0$$

for $\mathbb{E}_1^0 = \{u \in \mathbb{E}_1 \mid u(0) = 0\}$. A standard Neumann series argument shows that a whole ball exists around this operator in the norm of $\mathcal{L}(\mathbb{E}_1^0, \mathbb{E}_0)$ containing only invertible operators. Looking for $u = u_0 + v$, the nonlinear equation can be rewritten as

$$\dot{v} = F(u_0 + v),\ v(0) = 0.$$

Consider next the parameter dependent problem

$$\dot{v} = sF(u_0 + v) + (1 - s)DF(u_0)v,\ v(0) = 0,$$

where $s \in [0, 1]$, and the associated function $G(s, v) = \dot{v} - sF(u_0 + v) - (1 - s)DF(u_0)v$. Clearly this problem has the trivial solution as its unique solution on $[0, T]$ for $s = 0$. It also holds that

$$\begin{aligned} D_v G(s, v) &= \partial_t - sDF(u_0 + v) - (1 - s)DF(u_0) \\ &= \partial_t - DF(u_0) - s\big[DF(u_0 + v) - DF(u_0)\big], \end{aligned}$$

so that $D_v G(0, 0)$ is invertible. The implicit function theorem yields the existence of $s_0 > 0$ for which the parameter dependent problem has a unique solution $v(s)$ for each $s \in [0, s_0]$. If $s_0 \geq 1$, we are done. If not, it is possible to choose a smaller time $T_1 > 0$ so that

$$\begin{aligned} &\big\| D_v G\big(s_0, v(s_0)\big) - [\partial_t - DF(u_0)]\big\|_{\mathcal{L}(\mathbb{E}_1^0, \mathbb{E}_0)} \\ &\leq \big\| DF\big(u_0 + v(s_0)\big) - DF(u_0)\big\|_{\mathcal{L}(\mathbb{E}_1^0, \mathbb{E}_0)} << 1, \end{aligned}$$

thanks to the continuity of DF, provided $\|v(s_0)\|_{\mathrm{L}^\infty([0,T_1],E_1)} << 1$. Latter is clearly possible since $v(s_0)$ vanishes in $t = 0$. Notice that the inequality is independent of $s \in [0, 1]$. Thus, repeating the argument finitely many times, it is possible to get to $s = 1$ at the expense of possibly reducing the interval length to a final size of $T_0 > 0$. □

Remark 5.7 Given $\mathbb{F} \in \mathrm{C}^1(\mathbb{E}_0 \times \mathbb{E}_1, \mathbb{E}_0)$, it is easy to generalize the above theorem to the equation

$$\begin{cases} \mathbb{G}(u) = \mathbb{F}(\dot{u}, u) = 0, \\ u(0) = u_0 \in E_1, \end{cases}$$

provided that it be assumed that $D\mathbb{G}(\mathbb{u}_0) = D_1\mathbb{F}(0, \mathbb{u}_0)\partial_t + D_2\mathbb{F}(0, \mathbb{u}_0)$ is invertible (has maximal regularity) for $\mathbb{u}_0 \in \mathbb{E}_1$ with $\mathbb{u}_0(t) = u_0$ for all t.

Acknowledgements The author would like to thank the anonimous referees for the careful reading of the manuscript and for their valuable comments.

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