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Equilibria and their stability for a viscous droplet model

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Abstract

A classical model of fluid dynamics is considered which describes the shape evolution of a viscous liquid droplet on a homogeneous substrate. All equilibria are characterized and their stability is analyzed by a geometric reduction argument.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Consider the shape of a droplet of a viscous liquid on a homogeneous substrate and denote by $\Omega(t)$ the region wetted by the liquid. The droplet can then be described by a height field $u : \Omega(t) \to \mathbb{R}$ at any given time during the evolution (at least in the regime of interest). Asymptotic and averaging techniques yield, on appropriate assumptions (very small and very viscous droplet, see [8]), a simplification of Navier–Stokes equations which is considered here. The system in question reads

$$\begin{cases} -\Delta u = \lambda & \text{in } \Omega(t), \text{ for } t > 0, \\ u = 0 & \text{on } \partial\Omega(t), \text{ for } t > 0, \\ \int_{\Omega(t)} u \, \mathrm{d}x = V_0 > 0 & \text{for } t > 0, \\ V = F(|Du|) & \text{on } \partial\Omega(t), \text{ for } t > 0, \\ \Omega(0) = \Omega_0. \end{cases} \tag{1.1}$$

The third equations encodes conservation of volume and is responsible for the presence of the time dependent and unknown parameter λ in the first equation (details can again be found in [8]). The nonlinearity F drives the evolution, i.e. the contact angle dynamics, via the fourth equation in (1.1) for the front velocity V in outward normal direction $\nu(t)$ to the surface $\Gamma(t) = \partial\Omega(t)$. It is easily seen that balls evolve invariantly (maintaining their shape) for these equations and that a radius r_e is singled out by the volume conservation constraint combined with any canonical choice of F such as

$$F(s) = s^2 - 1 \text{ or } F(s) = s^3 - 1, \ s > 0,$$

and that such equilibrium ball is stable for the corresponding ordinary differential equation it satisfies (see [8]). For the purposes of this paper it will only be assumed that

$$F' > 0 \text{ and that } F(1) = 0, \tag{1.2}$$

thus effectively prescribing the contact angle below which the droplet would tend to locally retract and above which it would locally expand. While model (1.1) yields droplets with concave height profile, it is experimentally observed that the actual droplet pofile becomes convex before contact (the outermost boundary of the wetted region) and might exhibit a precursor film beyond contact, in which case latter would be defined as the location where the change in convexity occurs. Detailed analysis of the fluid behavior in the vicinity of contact leads to different effective driving forces F (see [4, 15] for instance). Assumptions (1.2) can be viewed as minimal qualitative assumptions of any physically relevant model. They are motivated by [5], where the author states that *It is always found that* $\frac{\partial\theta}{\partial U} \geqslant 0$ *whether or not the other fluid is a gas or an immiscible liquid* when talking about the shape evolution of a fluid droplet located on a substrate and surrounded by another fluid or a gas. In the notation used therein this means that the velocity of the wetting front always increases as a function of the contact angle.

It will be shown here that circles are the only equilibria of (1.1) and that they are stable with respect to any smooth perturbation. In particular, a perturbed circle will converge exponentially fast back to a circle albeit centered at a possibly different point. Its radius is, however, uniquely determined by volume conservation and therefore remains unchanged as compared to that of the circle being perturbed. The proof of stability hinges on the explicit computation of the linearization of (1.1) in a circle and on the use of a special nonlinear coordinate system in the 'space of curves' suggested by the translation invariance of the system.

Previous results about this basic model of fluid dynamics include local and global existence of appropriate weak solutions [10, 11] which would cover instances where singularity formation can occur (see the numerical experiments of [7]) and a local well-posedness result [6] in the category of classical solutions.

The remainder of the paper is organized as follows. In the next section the results of [6] are briefly summarized in order to introduce the appropriate functional setup and since they form the starting point for the subsequent analysis. Then equilibria are characterized and their stability is investigated by a fully explicit calculation of the linearization of the nonlinear, nonlocal curve evolution described by the last two equations of (1.1) and obtained by thinking of the other unknown, i.e. u, as the function of $\Gamma(t)$ determined by solving the first three equations, combined with the introduction of convenient nonlinear coordinates in the 'space of curves' Γ with respect to which the linearization in the equilibrium circle coincides with the computed one. In these nonlinear coordinates the evolution admits a simplified description of the dynamics obtained by a direct and revealing exploitation of the translation invariance of the system.

2. Setup

In order to reduce system (1.1) to an evolution equation for a simple unknown, an appropriate parametrization of $\Gamma(t)$ is necessary. To that end, fix a C^∞-hypersurface Γ close to $\Gamma_0 = \partial\Omega(0)$ in such a way that the latter can be described as a graph in normal direction over Γ, i.e.

$$\Gamma_0 = \{x + \rho_0(x)\nu(x) \,|\, x \in \Gamma\},$$

for some function $\rho_0 : \Gamma \to \mathbb{R}$. For technical reasons (better invariance properties with respect to interpolation) so-called little Hölder spaces prove a convenient choice of phase space. It is recalled that, for $\alpha \in (0,1)$ and an open subset of $\mathbb{R}^n$, the space of bounded uniformly Hölder continuous functions of exponent α

$$\mathrm{BUC}^\alpha(O) = \left\{ u : O \to \mathbb{R} \,|\, \|u\|_\infty < \infty,\ [u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty \right\}$$

is a Banach space with respect to the norm

$$\|\cdot\|_{\alpha,\infty} = \|\cdot\|_\infty + [\cdot]_\alpha.$$

For $k \in \mathbb{N}$ one also defines

$$\mathrm{BUC}^{k+\alpha}(O) = \{u \in \mathrm{BUC}^k(O) \,|\, \partial^\beta u \in \mathrm{BUC}^\alpha(O)\ \forall\, |\beta| = k\},$$

which is a Banach space with respect to the norm $\|\cdot\|_{k,\alpha} = \|\cdot\|_{k,\infty} + \sup_{|\beta|=k}[\partial^\beta \cdot]_\alpha$. Here it is used that

$$\mathrm{BUC}^k(O) = \{u : O \to \mathbb{R} \,|\, u \text{ is } k-\text{timescontinuously differentiable with bounded and uniformly continuous derivatives}\}$$

and $\|\cdot\|_{k,\infty} = \sup_{|\beta|\leqslant k}\|\partial^\beta u\|_\infty$. Then the little Hölder spaces are given by

$$h^{k+\alpha}(O) = \mathrm{closure}_{\|\cdot\|_{k,\alpha}}[r_O \mathcal{S}(\mathbb{R}^n)],$$

i.e. as the completion of the restriction r_O of smooth rapidly decreasing functions $\mathcal{S}$ to the set O with respect to the $\mathrm{BUC}^{k+\alpha}$ topology. These spaces can all be transplanted on any smooth compact manifold by the use of standard localization techniques and a smooth partition of unity. This is how the notation $h^{k+\alpha}(\Gamma)$ should be interpreted. Any given function $\rho \in h^{2+\alpha}(\Gamma)$ yields a diffeomorphism θ_ρ between Γ and $\Gamma_\rho = \{x + \rho(x)\nu(x) \,|\, x \in \Gamma\}$, which can be extended to a diffeomorphism of $\mathbb{R}^n$ still denoted by θ_ρ such that

$$\theta_\rho : \mathbb{R}^n \to \mathbb{R}^n,\ y \mapsto \begin{cases} X(y) + [\Lambda(y) + \varphi(\Lambda(y))\rho(X(y))]\nu(X(y)), & y \in \Omega_\Lambda, \\ y, & y \notin \Omega_\Lambda, \end{cases}$$

where φ is a smooth cut-off function, Ω_Λ is a tubular neighborhood of Γ and $(X(y), \Lambda(y)) \in \Gamma \times (-\Lambda, \Lambda)$ are 'tubular coordinates' of y, i.e. they satisfy

$$y = X(y) + \Lambda(y)\nu(X(y)).$$

This set up is visulalized in figure 1 for the benefit of the reader. Then

$$\theta_\rho(\Omega) = \Omega_\rho,\ \theta_\rho(\Gamma) = \Gamma_\rho,\ \theta_\rho|_{\Omega_\Lambda^c} = \mathrm{id}.$$

Clearly the tubular neighborhood is taken as small as the geometry of Γ requires in order to obtain a well-defined coordinate system $(X(y), \Lambda(y))$ and ρ small enough as to ensure that $\Gamma_\rho \subset \Omega_\Lambda$. More explicit and quantitative assumptions can be found in [6] but are not needed in

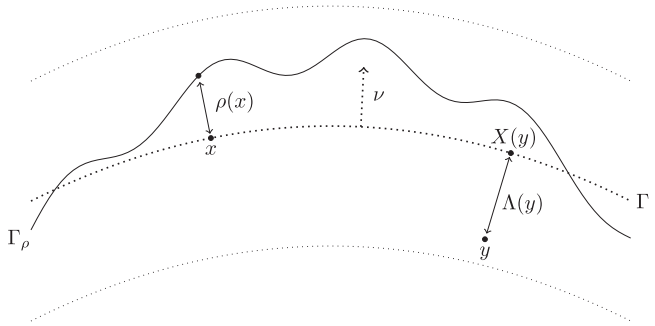

Figure 1. The local picture in a tubular neighborhood, bounded by the upper and lower most dotted curves, of the reference manifold Γ. The domain Ω_ρ is bounded from above by Γ_ρ.

the remainder of this paper. It should, however, be observed that the smallness assumption on $\rho \in h^{2+\alpha}(\Gamma)$ is immaterial since $\Gamma \in C^\infty$ can be chosen arbitrarily close to Γ_0 so that ρ_0 will be small. In the analysis to follow this represents no restriction. With the diffeomorphism θ_ρ in hand, the first three equations of (1.1) can be pulled back to a fixed domain Ω to give

$$\begin{cases} \mathbb{A}(\rho)v = -\theta_\rho^* \Delta \theta_*^\rho v = \lambda & \text{in } \Omega, \text{ for } t>0, \\ v = 0 & \text{on } \Gamma, \text{ for } t>0, \\ \displaystyle\int_\Omega v|\det D\theta_\rho| \, \mathrm{d}y = V_0 > 0 & \text{for } t>0, \end{cases} \tag{2.1}$$

by means of

$$\theta_\rho^* : h^{2+\alpha}(\Omega_\rho) \to h^{2+\alpha}(\Omega), \ u \mapsto v = u \circ \theta_\rho,$$

and the associated push-forward θ_*^ρ given by its inverse. This can be done on the assumption that $\Omega(t) = \Omega_{\rho(t,\cdot)}$ for a given

$$\rho : [0,T] \to h^{2+\alpha}(\Gamma),$$

yielding $\Gamma(t) = \Gamma_{\rho(t,\cdot)}$ and satisfying

$$\rho(t,\cdot) \in \mathbb{B}_{h^{1+\alpha}(\Gamma)}(0,\delta) \cap h^{2+\alpha}(\Gamma) =: \mathcal{V} \text{ for } t \in [0,T],$$

for a small $\delta > 0$ as described in [6]. Notice that $\mathbb{B}_X(0,\delta)$ represents the ball of radius δ about 0 in the space X. It is useful to denote the solution of (2.1) obtained by fixing $\rho \in \mathcal{V}$ by

$$v_\rho = \theta_\rho^* u_\rho = \mathbb{S}(\rho)\mathbf{1},$$

where $\mathbb{S}(\rho)\mathbf{1} = \lambda(\rho)\bar{\mathbb{S}}(\rho)\mathbf{1}$ for the solution $\bar{\mathbb{S}}(\rho)\mathbf{1} = \bar{v}_\rho = \theta_\rho^* \bar{u}_\rho$ of

$$\begin{cases} \mathbb{A}(\rho)v = 1 & \text{in } \Omega \text{ for } t>0, \\ v = 0 & \text{on } \Gamma \text{ for } t>0, \end{cases} \text{ or, equivalently, } \begin{cases} -\Delta u = 1 & \text{in } \Omega_\rho \text{ for } t>0, \\ u = 0 & \text{on } \Gamma_\rho \text{ for } t>0, \end{cases}$$

in the original coordinates, and where

$$\lambda(\rho) = \frac{V_0}{\displaystyle\int_\Omega \bar{v}_\rho |\det D\theta_\rho| \, \mathrm{d}y} = \frac{V_0}{\displaystyle\int_{\Omega_\rho} \bar{u}_\rho \, \mathrm{d}x},$$

rescales the solution to satisfy the volume constraint. As shown in [6], it is observed in passing that operators or other quantities depending on ρ, do indeed depend analytically on it. This fact will be needed later. In order to reformulate the kinematic equation in (1.1), it is convenient to define

$$N_\rho : \Omega_\Lambda \to \mathbb{R},\ y \to \Lambda(y) - \rho(X(y)),$$

so that

$$\Gamma_\rho = N_\rho^{-1}(0) \text{ and } \nu_\rho(y) = \frac{\nabla N_\rho(y)}{|\nabla N_\rho(y)|},\ y \in \Gamma_\rho.$$

Then it can be written

$$V(y,t) = \frac{\rho_t}{|\nabla N_\rho(y)|} = F(|Du_\rho|).$$

Observing that $|Du| = -\partial_\nu u$ on Γ_ρ in this particular case, the following single scalar nonlinear, nonlocal evolution equation for the 'shape function' ρ results

$$\begin{cases} \rho_t = |\nabla N_\rho| F(-\partial_{\nu_\rho} u_\rho) = |\nabla N_\rho| F(-\lambda(\rho)\partial_{\nu_\rho}\bar{u}_\rho) =: G(\rho) & \text{on } \Gamma \text{ for } t > 0, \\ \rho(0) = \rho_0 & \text{on } \Gamma. \end{cases} \tag{2.2}$$

It follows from [6] that (2.2) is equivalent to the original problem in the context of classical solutions, that G depends analytically on ρ and that

Theorem 2.1. *Given $\rho_0 \in \mathcal{V}$, there exists $T > 0$ and a unique solution*

$$\rho \in \mathrm{C}^1([0,T], h^{1+\alpha}(\Gamma)) \cap \mathrm{C}([0,T], h^{2+\alpha}(\Gamma))$$

of (2.2) and, thus, a solution $(\mathbb{S}(\rho)\mathbf{1}, \Omega_\rho)$ of (1.1). □

The proof relies on localization, perturbation, and optimal regularity results for parabolic equations which make it possible to reduce local well-posedness to properties of the linearization $DG(\rho_0)$ in the initial datum. The abstract approach of [6] is necessary to deal with the most general case but does not provide any explicit representation for the linearization. It is qualitative in nature and only yields that $DG(\rho_0)$ is the infinitesimal generator of an analytic semigroup on $h^{1+\alpha}(\Gamma)$ with domain $h^{2+\alpha}(\Gamma)$. Here and in order to obtain the stability results outlined earlier a much more detailed understanding of $DG(\rho_0)$ is necessary in the special case when $\Gamma = \Gamma_0 = \mathbb{S}_{r_e}$.

3. Equilibria

While the results remain valid for any space dimension, the analysis would have to be adapted to the specific dimension considered. The technique would essentially coincide in all dimensions but the specific spherical functions involved would have to be chosen depending on the dimension. In order to avoid rendering the presentation unnecessarily cumbersome, the choice is made to consider the case $n = 2$ in this paper.

An equilibrium solution of (1.1) is obtained if (u_e, Ω_e) can be found such that

$$V = F(|Du_e|) = 0.$$

On the assumptions made earlier this is the case only if

$$\partial_\nu u_e = -|Du_e| \equiv -1 \text{ on } \Gamma_e,$$

so that u_e satisfies

$$\begin{cases} -\Delta u_e = \lambda, & \text{in } \Omega_e, \\ u_e = 0, & \text{on } \Gamma_e, \\ \partial_\nu u_e = -1, & \text{on } \Gamma_e. \end{cases}$$

A classical rigidity result by Serrin [14] then implies that a classical solution of the above overdetermined system can only exist if Ω_e is a circle of some radius r_e determined by the additional requirement that

$$\int_{\Omega_e} u_e \, dx = V_0.$$

Theorem 3.1. *If (u_e, Ω_e) is a steady-state of (1.1), then Ω_e* must be a circle $\mathbb{B}_{r_e}$ *of radius*

$$r_e = \sqrt[3]{\frac{4V_0}{\pi}} \text{ and } u_e = \frac{1}{2}\left(r_e - \frac{r^2}{r_e}\right),$$

where r is the distance from the center of the circle. The parameter λ satisfies $\lambda = 2/r_e$.

Proof. Serrin's classical result implies that Ω_e is a circle. The rest follows from a direct computation. □

Remarks 3.2. (a) The author of [8] includes a partial stability result. He fixes a center of the circle and derives an ode describing the evolution of a circle of initial radius $r_0 \neq r_e$ with the same center. He shows that the circle of radius r_e is locally asymptotically stable.

(b) In order to obtain a stronger stability result the center and, more in general, the geometry needs to be perturbed as well. The 'freedom in the choice of center' is responsible for the existence of a (translational) eigenvalue $0 \in \sigma(DG(\rho_e))$ where $\rho_e \equiv 0$ when the reference manifold is the boundary $\mathbb{S}_{r_e}$ of the stationary solution itself.

(c) The functions in the kernel of $DG(0)$ can be computed by parametrizing the shifted circle

$$\partial\Omega_{\epsilon v} = \mathbb{S}_{r_e} + \epsilon z, \ z \in \mathbb{R}^2,$$

over the stationary reference circle which can be assumed to be centered in the origin without loss of generality. Using the representation described in the introduction through a parametrization over the reference circle amounts to determining the function $h_\epsilon : \mathbb{S} \to \mathbb{R}$ such that

$$\mathbb{S}_{r_e} + \epsilon z = \{[r_e + h_\epsilon(\phi)]\nu_e(\phi) \,|\, \phi \in [0, 2\pi)\},$$

where

$$\nu_e(\phi) = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}, \ \tau_e(\phi) = \begin{bmatrix} -\sin(\phi) \\ \cos(\phi) \end{bmatrix}$$

are the unit outward normal and unit tangent to the circle $\mathbb{S}_{r_e}$, respectively. This notation will be used again later. It must hold that

$$|[r_e + h_\epsilon(\phi)]\, \nu_e(\phi) - \epsilon z|^2 = r_e^2,$$

which, after differentiation in ϵ and evaluation in $\epsilon = 0$, yields

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} h_\epsilon(\phi) = \nu_{\mathrm{e}}(\phi)\cdot z = z_1\cos(\phi) + z_2\sin(\phi).$$

This shows that $z_1\cos + z_2\sin \in \ker(DG(0))$ for any $z \in \mathbb{R}^2$.

(d) It will be shown that the kernel only consists of the above 'translational' eigenvectors. Translation also yields a manifold of equilibria $\mathcal{E}$ (the set of all circles with fixed radius r_{e}) which can be locally parametrized as in (c). It in fact corresponds to a local center manifold for the evolution equation.

4. Linearization

Consider now the equilibrium $\mathbb{S}_{r_{\mathrm{e}}}$ centered at the origin and use it as the reference manifold Γ so that the equilibrium solution ρ_{e} vanishes identically. Then

$$X(y) = r_{\mathrm{e}}\frac{y}{|y|},\ \Lambda(y) = |y| - r_{\mathrm{e}},\ \nu_{\mathrm{e}} = \frac{y}{|y|},$$

and $y = X(y) + \Lambda(y)\frac{y}{|y|}$. In this case the function N_ρ is simply given by

$$N_\rho(y) = |y| - r_{\mathrm{e}} - \rho(y/|y|).$$

For ease of computation and notation the Euclidean coordinate y or the polar (r,ϕ) will be used interchangeably. In particular, functions on $\mathbb{S}_{r_{\mathrm{e}}}$ will be identified with functions of the angle variable ϕ. With this convention one has

$$\nabla N_\rho = \nu_{\mathrm{e}}(\phi) - \frac{\rho'(\phi)}{r_{\mathrm{e}}}\tau_{\mathrm{e}}(\phi) \text{ and } |\nabla N_\rho|^2 = 1 + r_{\mathrm{e}}^{-2}\rho'(\phi)^2.$$

Noticing that $|\nabla N_{\epsilon h}||_{\epsilon=0} = 1$ and recalling that $F(1) = 0$ it is arrived at

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} G(\epsilon h) = -F'(1)\frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}[\lambda(\epsilon h)\partial_{\nu_{\epsilon h}}\bar{u}_{\epsilon h}]. \tag{4.1}$$

It turns out, contrary to the approach taken in [6], that it is more convenient not to perform the transformation to a fixed domain when computing the linearization in a circle.

Theorem 4.1. *Let $h \in h^{2+\alpha}(\Gamma)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \partial_{\nu_{\epsilon h}}\bar{u}_{\epsilon h} = \frac{4V_0}{\pi r_{\mathrm{e}}^4}[r_{\mathrm{e}}DTN_{\mathbb{S}_{r_{\mathrm{e}}}}(h) - h],$$

where $DTN_{\partial\Omega}$ is the so-called Dirichlet-to-Neumann operator, i.e. the operator mapping a Dirichlet datum h to the outward normal derivative $\partial_\nu w_h$ of the solution w_h of the boundary value problem

$$\begin{cases} -\Delta w = 0, & \text{in } \Omega, \\ w = h, & \text{on } \partial\Omega. \end{cases}$$

Remark 4.2. The above theorem provides a formula for the first 'domain' variation of the solution of

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

in the circle of radius r_e, i.e. $\frac{\partial}{\partial\Omega}u\big|_{\Omega=\mathbb{B}_{r_e}}$.

Proof. Consider the solution $\bar{u}_{\epsilon h}$ of (4.2) for $\Omega=\Omega_{\epsilon h}$ and observe that $\bar{u}_0=\bar{u}_e$. If one looks for $\bar{u}_{\epsilon h}$ in the form

$$\bar{u}_{\epsilon h}=w_{\epsilon h}+\frac{1}{2}(r_e^2-|x|^2).$$

Then clearly

$$-\Delta\bar{u}_{\epsilon h}=-\Delta w_{\epsilon h}+1=1 \text{ or } -\Delta w_{\epsilon h}=0,$$

and

$$0=\bar{u}_{\epsilon h}\big|_{\Gamma_{\epsilon h}}=w_{\epsilon h}\big|_{\Gamma_{\epsilon h}}+\frac{1}{2}(r_e^2-(r_e+\epsilon h)^2) \text{ or } w_{\epsilon h}\big|_{\Gamma_{\epsilon h}}=\epsilon r_e h+\frac{\epsilon^2}{2}h^2.$$

It follows that

$$\partial_{\nu_{\epsilon h}}\bar{u}_{\epsilon h}=\partial_{\nu_{\epsilon h}}w_{\epsilon h}+\partial_{\nu_{\epsilon h}}\frac{1}{2}(r_e^2-|x|^2)=r_eDTN_{\Omega_{\epsilon h}}\left(\epsilon h+\frac{\epsilon^2}{2r_e}h^2\right)-(r_e+\epsilon h)\nu_{\epsilon h}\cdot\nu_e.$$

Since $\nu_{\epsilon h}=(-\epsilon h'\tau_e+(r_e+\epsilon h)\nu_e)/\sqrt{(r_e+\epsilon h)^2+\epsilon^2h'^2}$ one has that

$$\partial_{\nu_{\epsilon h}}\bar{u}_{\epsilon h}=r_eDTN_{\Omega_{\epsilon h}}\left(\epsilon h+\frac{\epsilon^2}{2r_e}h^2\right)-\frac{(r_e+\epsilon h)^2}{[(r_e+\epsilon h)^2+\epsilon^2h'^2]^{1/2}}.$$

Now, if DTN_{Ω_ρ} depends continuously on ρ, it can be easily inferred that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}\partial_{\nu_{\epsilon h}}\bar{u}_{\epsilon h}=r_eDTN_{\mathbb{B}_{r_e}}(h)-h.$$

The continuous dependence, however, follows from

$$DTN_{\Omega_\rho}(h)=\partial_{\nu_\rho}\bar{u}_\rho=\frac{\nabla N_\rho}{|\nabla N_\rho|}\cdot\nabla\bar{u}_\rho\big|_{\Gamma_\rho},$$

for

$$\bar{u}_\rho=-(\theta_\rho^*\Delta_{\mathbb{B}_{r_e}}\theta_*^\rho,\gamma_{\Gamma_\rho})^{-1}(1,0),$$

for $\rho\in\mathcal{V}$. It can be seen as in [6] that

$$[\rho\mapsto DTN_{\Omega_\rho}]:h^{2+\alpha}(\Gamma_\rho)\to\mathcal{L}(h^{2+\alpha}(\Gamma_\rho),h^{1+\alpha}(\Gamma_\rho))$$

is an analytic function because N_ρ and θ_ρ depend algebraically on ρ. Notice that $h:\mathbb{S}_{r_e}\to\mathbb{R}$, that is, a function of the angle variable ϕ only, can always be transplanted on Γ_ρ and identified with a function $\tilde{h}:\Gamma_\rho\to\mathbb{R}$ via

$$\tilde{h}([(r_e+\rho(\phi)]\nu_e(\phi))=h(\phi),\ \phi\in[0,2\pi).$$

Thus the operator DTN_{Ω_ρ} can be viewed as defined on the fixed space $h^{2+\alpha}(\mathbb{S}_{r_e}) \,\widehat{=}\, h_p^{2+\alpha}$, where the latter is the space of 2π-periodic little Hölder functions. □

In order to complete the evaluation of the linearization, the term

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \lambda(\epsilon h) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0} \frac{V_0}{\int_{\Omega_{\epsilon h}} \bar{u}_{\epsilon h}\,\mathrm{d}x}.$$

Needs to be evaluated. Thus consider

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \int_{\Omega_{\epsilon h}} \bar{u}_{\mathrm{e}}(x)\,\mathrm{d}x = \int_{\mathbb{S}_{r_e}} \bar{u}_{\mathrm{e}}(x)\,\mathrm{d}\sigma_{\mathbb{S}_{r_e}}(x) = 0,$$

by the boundary condition and where $\bar{u}_{\mathrm{e}} = \bar{u}_{\epsilon h}|_{\epsilon=0}$, so that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \int_{\Omega_{\epsilon h}} \bar{u}_{\epsilon h}(x)\,\mathrm{d}x = \int_{\mathbb{B}(0,r_e)} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \bar{u}_{\epsilon h}(x)\,\mathrm{d}x.$$

Using a Green's function representation for the solution, i.e.

$$\bar{u}_{\epsilon h}(x) = \int_{\Omega_{\epsilon h}} H_{\epsilon h}(x,\bar{x})\,\mathrm{d}\bar{x},$$

where $H_{\epsilon h}$ denotes the Green's function for the Dirichlet problem for the negative Laplacian on the domain $\Omega_{\epsilon h}$, this amounts to computing

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \bar{u}_{\epsilon h} &= \int_{\mathbb{S}_{r_e}} H_0(x,\bar{x})\,\mathrm{d}\sigma_{\mathbb{S}_{r_e}}(\bar{x}) + \int_{\mathbb{B}(0,r_e)} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} H_{\epsilon h}(x,\bar{x})\,\mathrm{d}\bar{x} \\ &= \int_{\mathbb{B}(0,r_e)} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} H_{\epsilon h}(x,\bar{x})\,\mathrm{d}\bar{x}.\end{aligned}$$

Notice that the boundary integral term vanishes because the Green's function is zero on the boundary. For the last term it is resorted to the so-called Hadamard domain variation formula [9] (see [13] for a generalized version and more recent developments) for Green's functions which, in this particular case, yields

$$H_{\epsilon h}(x,\bar{x}) - H_0(x,\bar{x}) = \epsilon \int_0^{2\pi} \partial_r H_0(x, r_{\mathrm{e}}, \phi)\partial_r H_0(\bar{x}, r_{\mathrm{e}}, \phi) h(\phi)\,\mathrm{d}\phi + o(\epsilon),$$

for any $h \in \mathrm{C}^{2+\alpha}(\mathbb{S}_{r_e})$. In order to continue the computation it is convenient to have an explicit formula for the Dirichlet Green's function H_0 for the circle of radius r_{e}

$$H_0(r,\theta,\bar{r},\bar{\theta}) = \frac{1}{4\pi}\log\left[\frac{r_{\mathrm{e}}^2 r^2 + r_{\mathrm{e}}^2\bar{r}^2 - 2r_{\mathrm{e}}^2 r\bar{r}\,\cos(\theta-\bar{\theta})}{r^2\bar{r}^2 + r_{\mathrm{e}}^4 - 2r_{\mathrm{e}}^2 r\bar{r}\,\cos(\theta-\bar{\theta})}\right]$$

from which it follows that

$$\partial_r H_0(r,\theta,r_{\mathrm{e}},\phi) = \frac{r_{\mathrm{e}}^2}{2\pi}\frac{r_{\mathrm{e}}^2 - r^2}{r_{\mathrm{e}}^2 r^2 + r_{\mathrm{e}}^4 - 2r_{\mathrm{e}}^3 r\,\cos(\theta-\phi)} = \frac{1}{2\pi}\frac{r_{\mathrm{e}}^2 - r^2}{r^2 + r_{\mathrm{e}}^2 - 2r_{\mathrm{e}} r\,\cos(\theta-\phi)}.$$

It is important to observe that the function

$$(r,\theta) \mapsto \int_0^{2\pi} \partial_r H_0(r,\theta,r_{\mathrm{e}},\phi) g(\phi)\,\mathrm{d}\phi$$

is harmonic in $\mathbb{B}(0,r_{\rm e})$ by choice and has boundary value g on $\mathbb{S}_{r_{\rm e}}$ since

$$\lim_{r\to r_{\rm e}}\partial_r H_0(r,\theta,r_{\rm e},\phi)=\delta(\theta-\phi).$$

Indeed the following identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\bar u_{\epsilon h}&=\int_{\mathbb{B}(0,r_{\rm e})}\frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}H_{\epsilon h}(x,\bar x)\,\mathrm{d}\bar x\\&=\int_0^{2\pi}\int_0^{r_{\rm e}}\int_0^{2\pi}\frac{1}{2\pi}\frac{r_{\rm e}^2-r^2}{r^2+r_{\rm e}^2-2r_{\rm e}r\,\cos(\theta-\phi)}\frac{1}{2\pi}\frac{r_{\rm e}^2-\bar r^2}{\bar r^2+r_{\rm e}^2-2r_{\rm e}\bar r\,\cos(\bar\theta-\phi)}h(\phi)\,\mathrm{d}\phi\bar r\mathrm{d}\bar r\mathrm{d}\bar\theta\\&=\int_0^{2\pi}\frac{1}{2\pi}\frac{r_{\rm e}^2-r^2}{r^2+r_{\rm e}^2-2r_{\rm e}r\,\cos(\theta-\phi)}\frac{r_{\rm e}^2}{2}h(\phi)\,\mathrm{d}\phi\end{aligned}$$

follows from the fact that

$$\int_0^{2\pi}\frac{r_{\rm e}^2-\bar r^2}{\bar r^2+r_{\rm e}^2-2r_{\rm e}\bar r\,\cos(\bar\theta-\phi)}\,\mathrm{d}\bar\theta\equiv 1,$$

as it is the unique harmonic function with constant value 1 on the boundary. Combining everything together it is arrived at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\int_{\Omega_{\epsilon h}}u_{\epsilon h}(x)\,\mathrm{d}x&=\int_{\mathbb{B}(0,r_{\rm e})}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\bar u_{\epsilon h}\,r\mathrm{d}r\mathrm{d}\theta\\&=\int_{\mathbb{B}(0,r_{\rm e})}\int_0^{2\pi}\frac{1}{2\pi}\frac{r_{\rm e}^2-r^2}{r^2+r_{\rm e}^2-2r_{\rm e}r\,\cos(\theta-\phi)}\frac{r_{\rm e}^2}{2}h(\phi)\,\mathrm{d}\phi\\&=\frac{r_{\rm e}^2}{2}\int_0^{2\pi}h(\phi)\int_0^{2\pi}\int_0^{r_{\rm e}}\frac{1}{2\pi}\frac{r_{\rm e}^2-r^2}{r^2+r_{\rm e}^2-2r_{\rm e}r\,\cos(\theta-\phi)}r\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi=\frac{r_{\rm e}^4}{4}\int_0^{2\pi}h(\phi)\,\mathrm{d}\phi=\frac{\pi r_{\rm e}^4}{2}\hat h_0,\end{aligned}$$

for the same reason as above. Here $\hat h_0$ is the average of the function h. Returning to the computation of the linearization it is seen that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\frac{V_0}{\int_{\Omega_{\epsilon h}}\bar u_{\epsilon h}\,\mathrm{d}x}=-\frac{V_0}{\left(\int_{\Omega_{\rm e}}\bar u_{\rm e}\,\mathrm{d}x\right)^2}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\int_{\Omega_{\epsilon h}}\bar u_{\epsilon h}\,\mathrm{d}x=-\frac{16V_0}{\pi^2r_{\rm e}^8}\frac{\pi r_{\rm e}^4}{2}\hat h_0=-\frac{8V_0}{\pi r_{\rm e}^4}\hat h_0.$$

This concludes the computation of the linearization which is summarized in the next theorem.

Theorem 4.3. *For $h\in h^{2+\alpha}(\mathbb{S}_{r_{\rm e}})$, it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}G(\epsilon h)=-F'(1)\left(\frac{4V_0}{\pi r_{\rm e}^4}[r_{\rm e}DTN_{\mathbb{S}_{r_{\rm e}}}(h)-h]+\frac{8V_0}{\pi r_{\rm e}^4}\hat h_0\right)\tag{4.3}$$

Proof. The calculations preceding the formulation of the theorem yield a complete proof by observing that (4.1) implies that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}G(\epsilon h)=-F'(1)\left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\lambda(\epsilon h)\partial_{\nu_{\rm e}}\bar u_{\rm e}+\lambda(0)\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\partial_{\nu_{\epsilon h}}\bar u_{\epsilon h}\right]$$

and also that $\partial_{\nu_e}\bar{u}_e \equiv -1$. □

Exploiting an alternative representation for the solution w_h of

$$\begin{cases} -\Delta w = 0, & \text{in } \mathbb{B}(0, r_e), \\ w = h, & \text{on } \mathbb{S}_{r_e}, \end{cases}$$

given by

$$w_h(r,\phi) = \frac{1}{2\pi}\hat{h}_0 + \sqrt{2}\sum_{k=1}^{\infty}\frac{r^k}{r_e^k}\Big[\hat{h}_k^c\cos(k\phi) + \hat{h}_k^s\sin(k\phi)\Big],\ (r,\phi)\in[0,r_e)\times[0,2\pi),$$

it is arrived at

$$DTN_{\mathbb{S}_{r_e}}(h) = \partial_{\nu_e}w_h = \frac{\sqrt{2}}{r_e}\sum_{k=1}^{\infty}k\Big[\hat{h}_k^c\cos(k\,\cdot) + \hat{h}_k^s\sin(k\,\cdot)\Big],$$

where $\hat{h}_k^c$, $\hat{h}_k^s$ are the Fourier coefficients of the function h with respect to the orthonormal basis

$$\frac{1}{2\pi}, \sqrt{2}\cos(\,\cdot), \sqrt{2}\sin(\,\cdot), \sqrt{2}\cos(2\,\cdot), \sqrt{2}\sin(2\,\cdot), \ldots$$

of $L^2(\mathbb{S}_{r_e})$. Together with representation (4.3) this yields

Theorem 4.4. *The spectrum of the linearization is given by*

$$\sigma(DG(0)) = -F'(1)\frac{4V_0}{\pi r_e^4}\{0,1,2,3,\ldots\}.$$

The kernel is precisely the two-dimensional space generated by cos and sin due to the translation invariance of the problem as observed in remarks 3.2(c) and (d). The first negative eigenvalue has eigenspace generated by the functions $\mathbf{1}, \cos(2\,\cdot), \sin(2\,\cdot)$, whereas the remaining negative eigenvalues corresponding to $k = 2, 3, \ldots$ have eigenspace generated by $\cos((k+1)\,\cdot), \sin((k+1)\,\cdot)$.

5. Stability analysis

For the purpose of analyzing the stability of equilibria it is more convenient to use a slightly different parametrization of the manifold of curves about a fixed steady-state. Any small enough $\rho \in \mathcal{V}$ can be described using the coordinates

$$(z(\rho), \bar{\rho}(\rho))$$

where $z \in \mathbb{R}^2$ is the 'spatial location coordinate' and $\bar{\rho}$ is the 'shape coordinate'. More precisely these coordinates are obtained from the identity

$$\Gamma_\rho = z + \Gamma_{\bar{\rho}}$$

by choosing the vector z so that $\bar{\rho} \in \ker(DG(0))^\perp$. Intuitively the function $\bar{\rho}$ fixes a geometric shape, while z moves it into its 'location'. It will be proved in lemma 5.2 that this is indeed a well-defined coordinate system. The main reason for its use is that the evolution equation takes on a particularly easy normal form, see system (5.6), in these coordinates due to the fact that

Lemma 5.1. *It holds that* $G(\rho(z,\bar\rho)) = G(\rho(0,\bar\rho)) = G(\bar\rho)$ *for all* $z\in\mathbb{R}^2$.

Proof. Given $\rho\in\mathcal{V}$ consider the domains Ω_ρ and $z+\Omega_\rho$ for $z\in\mathbb{R}^2$. The solution u_z of

$$\begin{cases} -\Delta u = \lambda & \text{in } z+\Omega_\rho,\\ u = 0 & \text{on } \Gamma_\rho,\\ \displaystyle\int_{\Omega_\rho} u(x)\,\mathrm{d}x = V_0, \end{cases}$$

clearly satisfies

$$u_z(x) = u_0(x-z),\ x\in z+\Omega_\rho$$

so that

$$\partial_{\nu_{z+\Gamma_\rho}} u_z = \partial_{\nu_{\Gamma_0}} u_0(\cdot - z).$$

If ϕ is the angle variable, then

$$\partial_{\nu_{z+\Gamma_\rho}} u_z(\phi) = \partial_{\nu_{\Gamma_0}} u_0(\phi),$$

and so $G(\rho) = G(\rho(z,\bar\rho)) = G(\bar\rho)$. □

In the next lemma, it is verified that $(z,\bar\rho)$ indeed is a well-defined coordinate system for $h^{2+\alpha}(\mathbb{S}_{r_e})$ about $\rho\equiv 0$.

Lemma 5.2. *For any given* $\rho\in\mathcal{V}$ *small enough, there is a unique small* $z\in\mathbb{R}^2$ *such that*

$$\Gamma_\rho = z+\Gamma_{\bar\rho},$$

for some small $\bar\rho\in N(DG(0))^\perp$.

Proof. For the two sets to coincide, given a point

$$[r_e+\rho(\theta)]\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}\in\Gamma_\rho$$

and a small $z\in\mathbb{R}^2$, $(\varphi,\bar\rho(\varphi))$ needs to be found such that

$$[r_e+\rho(\theta)]\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} - \begin{bmatrix}z_1\\ z_2\end{bmatrix} = [r_e+\bar\rho(\varphi)]\begin{bmatrix}\cos(\varphi)\\ \sin(\varphi)\end{bmatrix}.$$

This identity implies that

$$\bar\rho(\varphi) = -r_e + [(r_e+\rho)^2 + |z|^2 - 2(r_e+\rho)z\cdot\nu_e]^{1/2}, \tag{5.1}$$

$$\tan(\varphi) = \frac{(r_e+\rho)\sin(\theta) - z_2}{(r_e+\rho)\cos(\theta) - z_1}, \tag{5.2}$$

$$\cot(\varphi) = \frac{(r_e+\rho)\cos(\theta) - z_1}{(r_e+\rho)\sin(\theta) - z_2}, \tag{5.3}$$

where $\varphi = \varphi(\theta)$ and the dependence of ρ on θ is not explicitly indicated in order to present leaner formulæ. The freedom of choice between representations (5.2) and (5.3) will be exploited below. It is also useful to have

$$(1+\tan^2\varphi)\varphi'(\theta) = \frac{\rho' z\cdot\nu_{\mathrm{e}} + (r_{\mathrm{e}}+\rho)^2 - (r_{\mathrm{e}}+\rho)z\cdot\tau_{\mathrm{e}}}{[(r_{\mathrm{e}}+\rho)\cos(\theta)-z_1]^2}, \tag{5.4}$$

$$(1+\cot^2\varphi)\varphi'(\theta) = \frac{\rho' z\cdot\tau_{\mathrm{e}} + (r_{\mathrm{e}}+\rho)^2 - (r_{\mathrm{e}}+\rho)z\cdot\nu_{\mathrm{e}}}{[(r_{\mathrm{e}}+\rho)\sin(\theta)-z_2]^2}. \tag{5.5}$$

At this point only z needs to be determined. This is done by requiring that the following orthogonality conditions be satisfied

$$\begin{aligned}\Phi_\zeta(z_1,z_2) &= \int_0^{2\pi}\bar\rho(\varphi)\zeta(\varphi)\,\mathrm{d}\varphi = \int_{\varphi(0)}^{\varphi(2\pi)}\bar\rho(\varphi)\zeta(\varphi)\,\mathrm{d}\varphi\\ &= \int_0^{2\pi}\bar\rho(\varphi(\theta))\zeta(\varphi(\theta))\varphi'(\theta)\,\mathrm{d}\theta = 0 \text{ for } \zeta=\sin,\cos\,.\end{aligned}$$

Next the Jacobian of $\Phi = \begin{bmatrix}\Phi_{\cos}\\ \Phi_{\sin}\end{bmatrix}$ is computed and is shown to be non-singular for $\rho \ll 1$, which then implies the claim and concludes the proof. In order to compute $\partial_{z_1}\Phi$, representations (5.3)/(5.5) turn out to be more convenient and lead to

$$\begin{aligned}\partial_{z_1}\Phi_\zeta(0,0) = &-\int_0^{2\pi}\frac{1}{2}\frac{2(r_{\mathrm{e}}+\rho)\cos(\theta)}{r_{\mathrm{e}}+\rho}\frac{(r_{\mathrm{e}}+\rho)^2}{[1+\cot^2(\theta)](r_{\mathrm{e}}+\rho)^2\sin^2(\theta)}\zeta(\theta)\,\mathrm{d}\theta\\ &+\int_0^{2\pi}\rho\frac{(r_{\mathrm{e}}+\rho)^2}{[1+\cot^2(\theta)]^2}\frac{2\cos(\theta)}{(r_{\mathrm{e}}+\rho)^2\sin^3(\theta)}\frac{1}{(r_{\mathrm{e}}+\rho)\sin(\theta)}\zeta(\theta)\,\mathrm{d}\theta\\ &-\int_0^{2\pi}\rho\frac{1}{1+\cot^2(\theta)}\frac{\rho'\sin(\theta)+(r_{\mathrm{e}}+\rho)\cos(\theta)}{(r_{\mathrm{e}}+\rho)^2\sin^2(\theta)}\zeta(\theta)\,\mathrm{d}\theta\\ &+\int_0^{2\pi}\rho\frac{(r_{\mathrm{e}}+\rho)^2}{1+\cot^2(\theta)}\frac{1}{(r_{\mathrm{e}}+\rho)^2\sin^2(\theta)}\{\partial_{z_1}|_{(0,0)}\zeta(\varphi)\}\,\mathrm{d}\theta\\ = &\int_0^{2\pi}\left\{\left[-1+2\frac{\rho}{r_{\mathrm{e}}+\rho}\right]\cos(\theta)-\frac{\rho}{(r_{\mathrm{e}}+\rho)^2}[\rho'\sin(\theta)+(r_{\mathrm{e}}+\rho)\cos(\theta)]\right\}\zeta(\theta)\,\mathrm{d}\theta+\int_0^{2\pi}\rho(\theta)\zeta'(\theta)\,\mathrm{d}\theta,\end{aligned}$$

where the last term is obtained by observing that

$$\begin{aligned}\partial_{z_1}|_{(0,0)}\zeta(\varphi) &= \zeta'(\varphi)\varphi'|_{(0,0)}\\ &= \zeta'(\theta)\frac{(r_{\mathrm{e}}+\rho)^2}{[1+\cot^2(\theta)](r_{\mathrm{e}}+\rho)^2\sin^2(\theta)}\\ &= \zeta'(\theta),\end{aligned}$$

for $\theta\in[0,2\pi)$. It follows that

$$\begin{aligned}\partial_{z_1}\Phi_{\cos}(0,0) &= -1/2 + O(\|\rho\|_{h_p^{1+\alpha}}) \text{ as } \rho\to 0,\\ \partial_{z_2}\Phi_{\cos}(0,0) &= O(\|\rho\|_{h_p^{1+\alpha}}) \qquad\quad \text{as } \rho\to 0.\end{aligned}$$

In an analogous manner, but using representations (5.2)/(5.4) instead, it can be seen that

$$\partial_{z_1}\Phi_{\cos}(0,0) = O(\|\rho\|_{h_p^{1+\alpha}}) \qquad \text{as } \rho \to 0,$$
$$\partial_{z_2}\Phi_{\sin}(0,0) = -1/2 + O(\|\rho\|_{h_p^{1+\alpha}}) \text{ as } \rho \to 0,$$

so that the proof is complete. □

Theorem 5.3. *The nonlinear evolution equation (2.2) is equivalent to the system given by*

$$\begin{cases} \dot z = M(\bar\rho)^{-1}\begin{bmatrix} \pi_1^c G(\bar\rho) \\ \pi_1^s G(\bar\rho) \end{bmatrix} \\ \dot{\bar\rho} = \pi_1^\perp \widetilde G(\bar\rho), \end{cases} \tag{5.6}$$

where

$$\widetilde G(\bar\rho) = G(\bar\rho) - \frac{\bar\rho'}{r_{\mathrm e}+\bar\rho}\tau_{\mathrm e}\cdot M(\bar\rho)^{-1}\begin{bmatrix} \pi_1^c G(\bar\rho) \\ \pi_1^s G(\bar\rho) \end{bmatrix} = G(\bar\rho) + O(\|\bar\rho\|^2_{h_p^{1+\alpha}}).$$

and

$$M(\bar\rho) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sqrt{2}\begin{bmatrix} -\displaystyle\int_0^{2\pi}\frac{\bar\rho'}{r_{\mathrm e}+\bar\rho}\sin(\phi)\cos(\phi)\,\mathrm d\phi & \displaystyle\int_0^{2\pi}\frac{\bar\rho'}{r_{\mathrm e}+\bar\rho}\cos^2(\phi)\,\mathrm d\phi \\ \displaystyle\int_0^{2\pi}\frac{\bar\rho'}{r_{\mathrm e}+\bar\rho}\sin^2(\phi)\,\mathrm d\phi & \displaystyle\int_0^{2\pi}\frac{\bar\rho'}{r_{\mathrm e}+\bar\rho}\cos(\phi)\sin(\phi)\,\mathrm d\phi \end{bmatrix} \tag{5.7}$$

Proof. In the coordinates introduced above, one has that

$$V_\rho = \left(\dot z + \dot{\bar\rho}(\phi)\begin{bmatrix}\cos(\phi)\\ \sin(\phi)\end{bmatrix}\Big|\,\nu_\rho\right)$$

while

$$\nu_\rho = \frac{1}{\sqrt{(r_{\mathrm e}+\bar\rho)^2+\bar\rho'^2}}[(r_{\mathrm e}+\bar\rho)\nu_{\mathrm e} - \bar\rho'\tau_{\mathrm e}].$$

It follows that

$$\left[1+\left(\frac{\bar\rho'}{r_{\mathrm e}+\bar\rho}\right)^2\right]^{1/2} V_\rho = \dot z_1\left[\cos(\phi) - \frac{\bar\rho'}{r_{\mathrm e}+\bar\rho}\sin(\phi)\right] + \dot z_2\left[\sin(\phi) + \frac{\bar\rho'}{r_{\mathrm e}+\bar\rho}\cos(\phi)\right] + \dot{\bar\rho}. \tag{5.8}$$

Next, denoting by π_1^c, π_1^s, and $\pi_1^\perp$ the (orthogonal) projections onto $\mathbb{R}\cos(\phi)$, $\mathbb{R}\sin(\phi)$, and the orthogonal complement of $\mathbb{R}\cos(\phi)\oplus\mathbb{R}\sin(\phi)$, respectively, (5.8) entails that

$$\begin{cases} \dot z_1\Big\{1-\sqrt{2}\int_0^{2\pi}\frac{\bar\rho'}{r_{\rm e}+\bar\rho}\sin(\phi)\cos(\phi)\,{\rm d}\phi\Big\}+\dot z_2\Big\{\sqrt{2}\int_0^{2\pi}\frac{\bar\rho'}{r_{\rm e}+\bar\rho}\cos^2(\phi)\,{\rm d}\phi\Big\}=\pi_1^cG(\bar\rho),\\ \dot z_1\Big\{\sqrt{2}\int_0^{2\pi}\frac{\bar\rho'}{r_{\rm e}+\bar\rho}\sin^2(\phi)\,{\rm d}\phi\Big\}+\dot z_2\Big\{1+\sqrt{2}\int_0^{2\pi}\frac{\bar\rho'}{r_{\rm e}+\bar\rho}\cos(\phi)\sin(\phi)\,{\rm d}\phi\Big\}=\pi_1^sG(\bar\rho),\\ \dot{\bar\rho}=\pi_1^\perp G(\bar\rho)+\dot z_1\pi_1^\perp\Big(\frac{\bar\rho'}{r_{\rm e}+\bar\rho}\sin(\phi)\Big)-\dot z_2\pi_1^\perp\Big(\frac{\bar\rho'}{r_{\rm e}+\bar\rho}\cos(\phi)\Big). \end{cases}$$

Notice that the first two equations above amount to

$$M(\bar\rho)z=\begin{bmatrix}\pi_1^cG(\bar\rho)\\ \pi_1^sG(\bar\rho)\end{bmatrix}$$

for the matrix $M(\bar\rho)$ defined in (5.7), which satisfies

$$M(\bar\rho)=\begin{bmatrix}1&0\\0&1\end{bmatrix}+O(\|\bar\rho\|_{h_p^{1+\alpha}})\text{ as }\bar\rho\to0,$$

the claim follows by observing that it is invertible for small $\bar\rho$. □

Corollary 5.4. *Let $(u_{\rm e},\mathbb{B}(x_{\rm e},r_{\rm e}))\in\mathcal{E}$ be an equilibrium solution of (1.1). Then, for Γ_0 close enough to $\mathbb{S}_{r_{\rm e}}$, the solution $(u(\cdot,\Gamma_0),\Omega(\cdot,\Gamma_0))$ exists globally and there exists*

$$\rho(\cdot,\Gamma_0)\in{\rm C}^1\big([0,\infty),h_p^{1+\alpha}\big)\cap{\rm C}\big([0,\infty),h_p^{2+\alpha}\big)$$

with

$$\Omega(t,\Gamma_0)=\Omega_{\rho(t,\Gamma_0)}\text{ for }t\in[0,\infty),$$

as well as $z_\infty=z_\infty(\Gamma_0)\in\mathbb{R}^2$ such that

$$(z(\rho),\bar\rho(\rho))\longrightarrow(z_\infty,0)\text{ as }t\to\infty,$$

exponentially fast. In other words the manifold $\mathcal{E}$ of equilibria is locally asymptotically stable and any solution, starting close to it, converges exponentially fast to a specific $(u_\infty,\mathbb{B}(z_\infty,r_{\rm e}))$ which depends only on the initial condition.

Proof. Notice again that the vector field in (5.6) only depends on $\bar\rho$ and that, by construction,

$$\pi_1^\perp D\widetilde G(0)h=\pi_1^\perp DG(0)h=-F'(1)\frac{4V_0}{\pi r_{\rm e}^4}\Big(\hat h_0+\sum_{k\geqslant2}k[\hat h_k^{\,c}\cos(k\phi)+\hat h_k^{\,s}\sin(k\phi)]\Big),$$
$$h\in\pi_1^\perp h_p^{2+\alpha}=:h_{p,\perp}^{2+\alpha}.$$

It is not difficult to see that $\pi_1^\perp DG(0)\in\mathcal{H}(h_{p,\perp}^{2+\alpha},h_{p,\perp}^{1+\alpha})$, i.e the generator of an analytic semigroup on $h_{p,\perp}^{1+\alpha}$ with domain $h_{p,\perp}^{2+\alpha}$, either by applying [6, theorem 41] or by applying

Fourier multiplier results such as those found in [3] for 'periodic' symbols combined with a spectral reduction argument to split off the kernel, or by a direct computation of the associated semigroup and Fourier multiplier results. Observe that the subscript p is the notation of the function spaces indicates periodicity and stems from the standard identification

$$h^{k+\alpha}(\mathbb{S}_{r_c}) \mathrel{\widehat{=}} h_p^{k+\alpha},\ k\in\mathbb{N},$$

that was already used at the end of the proof of theorem 4.1.

It follows that the principle of linearized stability [12, theorem 9.1.2] applies and yields local asymptotic stability of the trivial solution $\bar\rho\equiv 0$ of $\dot{\bar\rho}=\pi_1^{\perp}\widetilde{G}(\bar\rho)$. Since the right-hand-side of (5.6) only depends on $\bar\rho$, since it is a smooth function of its argument, and since $G(0)=0$, it follows that

$$z(t)=z(0)+\int_0^t M(\bar\rho(\tau))^{-1}\begin{bmatrix}\pi_1^c G(\bar\rho(\tau))\\ \pi_1^s G(\bar\rho(\tau))\end{bmatrix}\mathrm{d}\tau\longrightarrow v_\infty \text{ as } t\to\infty.$$

The convergence is exponential since $\bar\rho$ converges to zero exponentially if $\bar\rho_0$ is small enough, which is always the case provided ρ_0 is. This same local analysis is valid about any other steady-state due to the translation invariance of the problem and the proof is complete. □

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