DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **5**, Number **3**, June **2012**

doi:10.3934/dcdss.2012.5.xx

pp. **X–XX**

A FAMILY OF NONLINEAR DIFFUSIONS CONNECTING PERONA-MALIK TO STANDARD DIFFUSION

Patrick Guidotti

Department of Mathematics
University of California
340 Rowland Hall
Irvine, CA 92697-3975, USA

Abstract. A one parameter family of equations is considered which connects the well-known Perona-Malik equation to standard diffusion. The parameter acts as a regularization parameter which gradually modifies the ill-posed Perona-Malik equation, through a strongly locally well-posed equation, to a strongly globally well-posed one which exhibits a behavior akin to that of standard diffusion, which is itself obtained in the limit. In the locally well-posed regime, the equation is degenerate parabolic and the onset of singularities can not be ruled out. Using a classical regularization approach, a-priori estimates can be derived which allow to go to limit with the regularization parameter globally in time. It is, however, not clear how to obtain a proper definition of weak solution for the limiting equation of interest.

1. Introduction.

In 1990 Perona and Malik proposed a new nonlinear diffusive model for noise reduction with applications to image processing. Equation

$$u_t - \operatorname{div}\big(g(|\nabla u|)\nabla u\big) = 0\,, \tag{1}$$

typically with $g(s) = \frac{1}{1+c^2s^2}$ has since become known as the Perona-Malik equation due to its similarity to the discrete equation found in the original [1]. The parameter c controls the size of gradients which one would like to preserve but has no significant influence on the mathematical properties of the equation and will therefore be chosen to be 1 in this paper. Equation (1) is an example of a *forward-backward* diffusion equation. This can be seen, on the one hand, by computing the divergence to find

$$u_t = g(|\nabla u|)\partial_{\tau\tau}u + \big[g(|\nabla u|) + 2|\nabla u|^2 g'(|\nabla u|)\big]\partial_{\nu\nu}u\,.$$

The coefficient of the derivative in direction ν normal to the level set of u becomes negative for large values of the argument ("edges" in an image) and typical choices of g as the one mentioned above. This leads to a regime where diffusion can reverse its sign at least in one direction. Alternatively the forward-backward character of the equation is reflected in the properties of the energy functional

$$E(u) = \frac{1}{2}\int_\Omega \log\big(1 + |\nabla u|^2\big)\,dx \tag{2}$$

2000 *Mathematics Subject Classification.* 35A01, 35A02, 35A09, 35K65, 35B40.

Key words and phrases. Nonlinear, nonlocal diffusion, Perona-Malik like equation, global existence.

This research was supported by the National Science Foundation under award number DMS-0712875.

of which (1) is the formal gradient flow. This is a convex-concave energy, the convexification of which is the zero functional. This feature makes the equation impermeable to methods of the calculus of variations and is at the origin of much of the interest it has generated and continues to generate in the mathematical community. In this paper focus will be primarily devoted to the one dimensional version of (1) and to equation

$$u_t - \frac{1}{1+|\triangle u|^2}\triangle u = 0\,, \tag{3}$$

which is, likewise, a forward-backward equation. Its one dimensional formulation can be obtained from (1) by rewriting the equation in terms of the new dependent variable $v = \int_0^{\cdot} u(\xi)\,d\xi$.

Equation (1) has steadily attracted the attention of the mathematical community since its introduction in 1990. A full satisfactory understanding of its properties still seems out of reach. A few direct attempts at its analysis have uncovered a variety of features it possesses [2, 3, 4, 5], some of which are somewhat unexpected [6]. In an alternative approach many attempts have been made to modify the equation with the goal of improving its mathematical properties, often with disregard for the practical implications for image processing which such modifications engender. Such attempts include [7, 8, 9, 10, 11, 12, 13] which range from spatial and/or temporal regularizations and relaxations of (1) to (spatially) semi-discrete models which are variational in character, mimic (1) and are more amenable to analysis.
Here the attention is focused on the regularizations proposed in [13] and partially analyzed in [4, 5]. They read

$$u_t - \operatorname{div}\big(\frac{1}{1+|\nabla^{1-\varepsilon}u|^2}\nabla u\big) = 0\,, \tag{4}$$

and

$$u_t - \frac{1}{1+|(-\triangle)^{1-\varepsilon}u|^2}\triangle u = 0\,, \tag{5}$$

where $\varepsilon \in (0,1]$. They distinguish themselves from other regularizations in two main respects. On the one hand they preserve an important dynamical feature of the original (1). Characteristic functions and continuous piecewise affine functions are indeed equilibria for equations (4) and (5), respectively. This was proved in [4, 5] and had been observed for (1) by [2, 14]. Other previous regularizations attempts such as [7] do cure the mathematical ills of (1) but also reduce to set of equilibria to trivial steady-states. The long term behavior of the associated dynamical system therefore becomes trivial. On the other hand the milder regularizations (4) and (5) turn the ill-posed (1) into a still degenerate equation for which global existence of smooth solutions is not a given. A nice transition from forward-backward diffusion, through degenerate diffusion, to standard plain linear-like diffusion will be shown to occur as ε goes from 0 to 1 for (5).
In [7] the regularization of (1) is implemented via convolution with a smooth kernel (a Gaussian G_σ) and, as a consequence, one has that

$$\|\nabla(G_\sigma u)\|_\infty \leq c(\sigma)\|u\|_\infty\,.$$

The nonlinear term $g(|\nabla(G_\sigma * u)|)$ therefore remains bounded for all times since the maximum principle ensures that

$$\|u(t)\|_\infty \leq \|u_0\|_\infty\,,$$

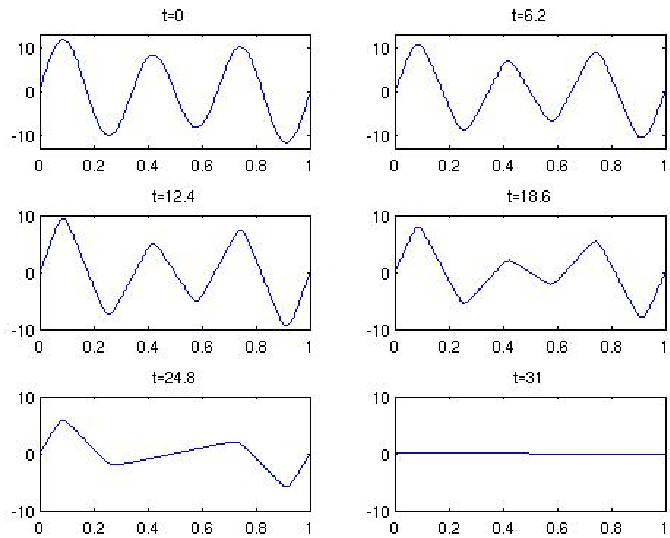

Figure 1. The typical evolution of a solution of (5) to a smooth initial datum for $\varepsilon < 1/2$. The evolution seems to follow a path that is, at all times, close to a non-trivial steady-state and to eventually, over the long run, converge to a trivial one.

for any solution of the regularized

$$u_t - \operatorname{div}\Big(\frac{1}{1+|\nabla(G_\sigma * u)|^2}\nabla u\Big) = 0 \tag{6}$$

originating in $u(0) = u_0 \in \mathrm{L}_\infty(\Omega)$. As regularizations (4) and (5) are obtained by convolution with a singular kernel they do not share this property with (6) and remain degenerate even for smooth initial data as singularity formation can not be ruled out in this case (at least for ε small). Numerical experiments clearly indicate that some kind of weak solution exists for all times. See Figures 1 and 2. Smooth solutions seem to evolve towards nontrivial singular steady-steady states and eventually, in the very long run, to a trivial equilibrium. As numerical diffusion might be playing a role, it is not obvious whether solutions really converge to a trivial steady-state or they develop singularities and converge to a non-trivial equilibrium or, still simply develop a singularity but eventually become constant. The same numerical experiments also suggest that, for equation (5), a transition to standard (linear) diffusive behavior occurs with fast convergence to trivial steady-states setting in as soon as $\varepsilon > 1/2$. This coincides with the threshold beyond which continuous piecewise affine functions cease to be equilibria for (5) unless they are constant. This paper is a first attempt at explaining and rigorously justifying these numerical findings by providing a solid analytical framework. Interesting open questions remain and will be formulated at the end of the paper.

In the next section the local existence results of [4, 5] are presented and adapted to secure existence of solutions for regularized versions of (4) and (5) which are then shown to be global in time. Subsequently a priori estimates are obtained

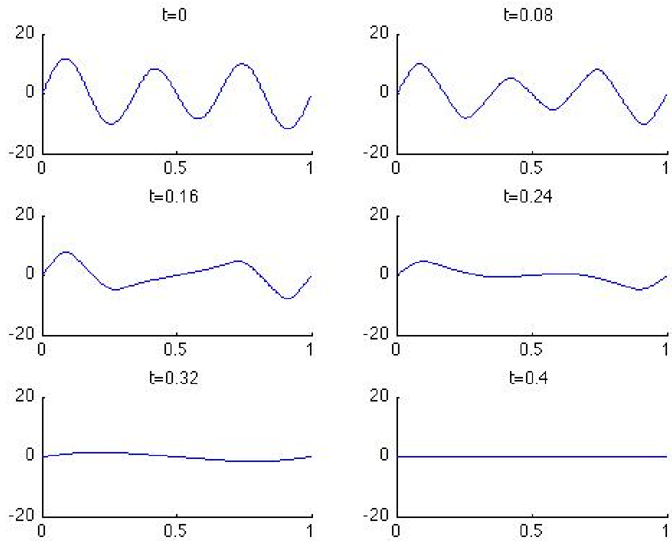

FIGURE 2. The typical evolution of a solution of (5) to a smooth initial datum for $\varepsilon > 1/2$. A solution converges very fast to a trivial equilibrium.

which are independent of the regularization parameter and which allow passage to the limit in appropriate topologies. Section 4 deals with the global existence of smooth solutions to (5) which is established for $\varepsilon > 1/2$ in perfect agreement with numerical experiments. In the concluding section global existence of a kind of weak solutions is discussed which, however, leaves many important questions unanswered.

2. **Local existence.** Local existence of smooth solutions for equations (4) and (5) has been obtained in [4] and [5], respectively using maximal regularity of parabolic problems and interpolation spaces to deal with the nonlocal nature of the nonlinearities involved. These results require a certain amount of regularity of the initial datum u_0. Namely one needs that $u_0 \in \mathrm{W}^s_p(\Omega)$ for $s > 1 - 2/p$ and $s > 2 - 2/p$ and for large enough $p > 1$ in order to obtain a local solution satisfying

$$u \in \mathrm{H}^1_p\big([0,T), \mathrm{W}^r_p(\Omega)\big) \cap \mathrm{L}_p\big([0,T), \mathrm{W}^{r-2}_p(\Omega)\big)\,,$$

for $r = 1, 2$, respectively. These results exploit the fact that, for smooth initial data, the equations cannot degenerate instantaneously and are never fully backward for any given $\varepsilon > 0$. For such $\varepsilon \in (0,1]$ the linearization of the equations at any smooth function has at most a finite number of growing modes in contrast to the infinite number of such unstable modes observed for supercritical initial data for the limiting equations.

For the purposes of this paper the following regularizations need also to be considered

$$u_t - \mathrm{div}\big([\delta + \frac{1}{1 + |\nabla^{1-\varepsilon} u|^2}]\nabla u\big) = 0\,, \tag{7}$$

and

$$u_t - [\delta + \frac{1}{1+|(-\triangle)^{1-\varepsilon}u|^2}]\triangle u = 0\,, \tag{8}$$

complemented with periodicity and initial conditions $u_0^\delta \in \mathrm{C}^\infty(\Omega)$ obtained by regularizing given functions $u_0 \in \mathrm{L}_\infty(\Omega)$ and $u_0 \in \mathrm{W}^1_\infty(\Omega)$ for (7) and (8), respectively.

Remark 1. While equations (4) and (5) are still degenerate, they allow for this kind of standard regularization, whereas the same technique can clearly not be used to regularize the limiting equations due to their "total" forward-backward nature.

The following theorem about local existence can be obtained by using the very same techniques of [4, 5] since the diffusion coefficients of the regularized equations satisfy the same (in fact better) assumptions.

Theorem 2.1. *Let $\varepsilon \in (0,1]$, $\delta > 0$, and assume u_0^δ is smooth as stated. Then equations (7) and (8) have a unique global smooth solution*

$$u \in \mathrm{C}^\infty\big((0,\infty), \mathrm{C}^\infty(\Omega)\big)\,.$$

The additional regularity follows by standard bootstrapping arguments once local existence is obtained in Sobolev spaces. Global existence is a consequence of the fact that the diffusion coefficient can never vanish due to the presence of the regularizing parameter. The regularized equations have Ljapunov functions given by the L_2-norm of the solution and of its gradient, respectively. This follows from

$$\frac{d}{dt}\int_\Omega |u(x)|^2\,dx = -2\int_\Omega |\nabla u|^2\big[\delta + \frac{1}{1+|\nabla^{1-\varepsilon}u|^2}\big]\,dx\,, \tag{9}$$

and

$$\frac{d}{dt}\int_\Omega |\nabla u(x)|^2\,dx = -2\int_\Omega |\triangle u|^2\big[\delta + \frac{1}{1+|(-\triangle)^{1-\varepsilon}u|^2}\big]\,dx\,, \tag{10}$$

respectively. In the stated form the theorem remains valid even if the initial conditions are not regularized because of the regularizing properties of (quasi-linear) parabolic flows.

3. **A priori estimates.** In order to establish the global existence of some kind of weak solution a-priori estimates need to be derived for the family of regularized solutions. The maximum principle is valid for smooth solutions of equations (7) and (8) and readily implies that

$$\|u^\delta(t,\cdot)\|_{\mathrm{L}_\infty(\Omega)} \le \|u_0^\delta\|_{\mathrm{L}_\infty(\Omega)} \text{ for any } t>0\,,$$

in both cases. Multiplying equation (7) by u^δ and equation (8), respectively, and integrating by parts , leads to (9) and (10). It easily follows that

$$\|u^\delta(t,\cdot)\|_{\mathrm{L}_2(\Omega)} \le \|u_0^\delta\|_{\mathrm{L}_2(\Omega)} \text{ for any } t>0\,,$$

which is no additional information, and

$$\|\nabla u^\delta(t,\cdot)\|_{\mathrm{L}_2(\Omega)} \le \|\nabla u_0^\delta\|_{\mathrm{L}_2(\Omega)} \text{ for any } t>0\,,$$

respectively. For equation (8) it is possible to obtain also some information about the Laplacian of the solution by a duality argument. Observing that $v = \triangle u$ satisfies

$$v_t = \triangle\big([\delta + \frac{1}{1+|(-\triangle)^{1-\varepsilon}u|^2}]v\big) \tag{11}$$

for the solution of (8) and that this equation is nothing but the formal adjoint of

$$\begin{cases} w_t = -a_\delta(t,x)\triangle w\,, \\ w(T) = w_T\,, \end{cases} \tag{12}$$

for $a_\delta(t,x) = \delta + \frac{1}{1+|(-\triangle)^{1-\varepsilon}u|^2}$ on any time interval $[0,T]$, it is possible to derive $\mathrm{L}_1(\Omega)$ estimates. Choose an end value w_T and multiply equation (11) by the corresponding solution. Integration by parts leads to

$$\int_\Omega v(T,x)w_T(x)\,dx = \int_\Omega v_0(x)w(0,x)\,dx\,.$$

and then to

$$\|v\|_{\mathrm{L}_1(\Omega)} = \sup_{\|w_T\|_\infty\leq 1}\int_\Omega v(T,x)w_T(x)\,dx = \sup_{\|w_T\|_\infty\leq 1}\int_\Omega v_0(x)w(0,x)\,dx\,.$$

As w satisfies (12), the maximum principle ensures that $\|w(0,\cdot)\|_\infty \leq \|w_T\|_\infty$ and thus

$$\|\triangle u\|_{\mathrm{L}_1(\Omega)} = \|v\|_{\mathrm{L}_1(\Omega)} \leq \|v_0\|_{\mathrm{L}_1(\Omega)}\,. \tag{13}$$

Remark 2. A similar estimate can not be derived for equation (7) in general spatial dimension. It is, however, valid in one space dimension since, then, an equation similar to (11) is satisfied by u_x as is easily checked.

All a priori estimates derived so far are clearly independent of $\delta > 0$ as long as appropriate assumptions are made on the initial conditions and provided such inequalities are respected when regularizing. In summary the following theorem is obtained.

Theorem 3.1. *Let $T > 0$ be given but arbitrary. Assume that $u_0 \in \mathrm{L}_\infty(\Omega)$ and let u^δ denote the solution of equation (7). Then $u^\delta(t,\cdot) \in \mathrm{L}_\infty\big([0,T],\mathrm{L}_\infty(\Omega)\big)$ uniformly in $\delta > 0$. If, in addition, $n = 1$ and $u_0 \in \mathrm{W}^1_1(\Omega)$, then*

$$u^\delta_x \in \mathrm{L}_\infty\big([0,T],\mathrm{L}_1(\Omega)\big) \text{ uniformly in } \delta > 0\,.$$

Next assume that $u_0 \in [\mathrm{L}_\infty \cap \mathrm{H}^1 \cap \mathrm{W}^2_1](\Omega)$ and let u^δ denote the solution of equation (8). Then

$$u^\delta \in \mathrm{L}_\infty\big([0,T],[\mathrm{L}_\infty \cap \mathrm{H}^1 \cap \mathrm{W}^2_1](\Omega)\big)$$

uniformly in $\delta > 0$.

The reason for the estimates being independent of $\delta > 0$ is simply that corresponding estimates remain valid in the limit as δ vanishes and, thus, the strong parabolicity induced by $\delta > 0$ does not enter. In the limit, no better estimates can be derived and thus a global weak solution has to be constructed resorting to these inequalities alone. Better inequalities can not possibly be derived in general; this is a consequence of the existence of non-trivial singular steady-states shown in [4, 5].

4. **Global existence of smooth solutions.** Numerical experiments strongly indicate that $\varepsilon = 1/2$ is a threshold value for equation (5) in one space dimension below which standard diffusion kicks in, smooth solutions exist globally and convergence to trivial equilibria is observed. For values of ε larger than this threshold value, singularity formation (even from smooth initial data) can not be excluded, non-trivial steady-states possibly determine the evolution and, in general, only a kind of weak solution can be shown to exist globally.

Lemma 4.1. *Let $\varepsilon > 1/2$. Then a constant $c \geq 0$ can be found such that*

$$\|(-\partial_{xx})^{1-\varepsilon} u\|_\infty \leq c \|\partial_x u\|_\infty \, . \tag{14}$$

Proof. Since $\varepsilon < 1/2$ the fractional Laplacian is a pseudo-differential operator or order strictly less than 1 and can therefore be viewed as a negative order operator acting on the first derivative. In the language of symbols

$$\begin{aligned}(-\partial_{xx})^{1-\varepsilon} &= \mathcal{F}^{-1} \operatorname{diag}\big[(2\pi k)^{2-2\varepsilon}\big] \mathcal{F} \\ &= \mathcal{F}^{-1} \operatorname{diag}\big[(-i(2\pi)^{1-2\varepsilon} \frac{k}{|k|^{2\varepsilon}})(-2i\pi k)\big] \mathcal{F} \\ &= \underbrace{\mathcal{F}^{-1} \operatorname{diag}\big[(-i(2\pi)^{1-2\varepsilon} \frac{k}{|k|^{2\varepsilon}})\big] \mathcal{F}}_{T_\varepsilon :=} \partial_x \end{aligned} \tag{15}$$

Classical results of harmonic analysis (see [15]) ensure that an operator T is bounded on $\mathrm{L}_\infty(\Omega)$ provided its symbol is the Fourier transform of a Borel measure. This is certainly the case for T_ε since the inverse Fourier transform of its symbol $-i(2\pi)^{1-2\varepsilon} \frac{k}{|k|^{2\varepsilon}}$ is an integrable function. Latter follows from the Poisson summation formula and known Fourier transforms (see again [15]). The claim is therefore a consequence of the factorization (15). □

With this inequality in hand it is now possible to prove

Theorem 4.2. *Let $n = 1$, $\varepsilon > 1/2$ and $u_0 \in \mathrm{W}^1_\infty(\Omega)$. Then the local smooth solution of (5) can be extended to a global smooth solution. It eventually converges to a trivial equilibrium.*

Proof. The function $v = u_x$ satisfies the equation

$$v_t = \big(\frac{1}{1 + |(-\partial_{xx})^{1-\varepsilon} u|^2} v_x\big)_x = \big(\frac{1}{1 + |T_\varepsilon v|^2} v_x\big)_x$$

and the maximum principle implies that $\|v\|_\infty \leq \|(u_0)_x\|_\infty$. In view of Lemma 4.1 the diffusion coefficient never vanishes since

$$\|T_\varepsilon v\|_\infty \leq c \|v\|_\infty \, ,$$

and the right-hand side of the latter remains bounded. As for the long time behavior, observe that (with (10))

$$\frac{d}{dt} \int_\Omega |u_x|^2 \, dx = -2 \int_\Omega \frac{1}{1 + |T_\varepsilon u_x|^2} u_{xx}^2 \, dx \leq -c \int_\Omega |u_{xx}|^2 \leq -c \int_\Omega |u_x|^2 \, dx$$

where the first inequality follows from the above estimate and the last from Poincaré's inequality. Convergence to trivial equilibria then follows as claimed. □

Remark 3. In [4] a different global existence proof (for smooth solutions) is given by other methods which is not optimal as it requires $\varepsilon > 3/4$. The result in this paper, while confirming the numerically observed threshold, leaves the question of global existence of smooth solution for $\varepsilon < 1/2$ open.

5. **Global existence of weak solutions.** In Section 3 the following δ-independent estimates

$$u^\delta \in \mathrm{L}_\infty([0,T]\times\Omega)\cap \mathrm{L}_\infty\bigl([0,T],\mathrm{H}^1(\Omega)\bigr)\cap \mathrm{L}_\infty\bigl([0,T],\mathrm{W}^1_2(\Omega)\bigr)$$

were established for solutions of (8) and for any arbitrary $T>0$. It follows that

$$\|u(t)-u(s)\|_{\mathrm{L}_1(\Omega)} = \|\int_s^t \dot u(\tau)\,d\tau\|_{\mathrm{L}_1(\Omega)} \leq \int_s^t \|\dot u(\tau)\|_{\mathrm{L}_1(\Omega)}\,d\tau$$
$$\leq \int_0^s \|\triangle u\|_{\mathrm{L}_1(\Omega)}\,d\tau \leq c(t-s)\,,\ 0\leq s\leq t\leq T\,, \quad (16)$$

where the constant c is independent of T and δ. Hence the family of solutions $\{u^\delta\,|\,\delta\in(0,1]\}$ is bounded in $\mathrm{C}^{1-}\bigl([0,T],\mathrm{L}_1(\Omega)\bigr)$. Interpolation and the a priori estimates then also yield

$$\|u(t)-u(s)\|_{\mathrm{L}_p(\Omega)} \leq \|u(t)-u(s)\|^{1/p}_{\mathrm{L}_1(\Omega)}\|u(t)-u(s)\|^{1/p'}_{\mathrm{L}_\infty(\Omega)} \leq c(t-s)^{1/p}\,,$$
$$0\leq s\leq t\leq T\,,$$

and $\{u^\delta\,|\,\delta\in(0,1]\}$ is also bounded in $\mathrm{C}^{1/p}\bigl([0,T],\mathrm{L}_p(\Omega)\bigr)$. A Banach space valued version of the Arzéla-Ascoli theorem then delivers a limiting function

$$u^0\in \mathrm{C}^\alpha\bigl([0,T],\mathrm{L}_{p,w}(\Omega)\bigr)$$

for any $\alpha<1/p$. The subscript w indicates that the space $\mathrm{L}_p(\Omega)$ is endowed with its weak topology. It clearly holds that

$$u^{\delta_j}\to u^0 \text{ in } \mathrm{C}^\alpha\bigl([0,T],\mathrm{L}_{p,w}(\Omega)\bigr)$$

for some $\delta_j\to 0$. This motivates the following definition

Definition 5.1. Let $u_0\in\mathrm{L}_\infty(\Omega)$. A function satisfying $u\in\mathrm{C}^\alpha\bigl([0,T],\mathrm{L}_{p,w}(\Omega)\bigr)$ for some $p>1$ and $\alpha<1/p$ is called *weak solution* for (5) if it can be obtained as the limit of a sequence u^{δ_j} of solutions of (8) for a null sequence $(\delta_j)_{j\in\mathbb{N}}$ and if

$$\lim_{t\to 0}\langle u(t,\cdot),\varphi\rangle = \langle u_0,\varphi\rangle\,,\ \varphi\in\mathrm{L}_{p'}(\Omega)\,.$$

With this definition it is easy to obtain the following result.

Theorem 5.2. *Equation* (5) *possesses a global weak solution*

$$u\in\mathrm{C}^\alpha\bigl([0,\infty),\mathrm{L}_{p,w}(\Omega)\bigr)$$

with $p\in(1,\infty)$ *and* $\alpha<1/p$ *to any given initial datum* $u_0\in\mathrm{W}^1_\infty(\Omega)\cap\mathrm{W}^2_1(\Omega)$.

Proof. The assumptions ensure that the all a priori estimates are valid for the regularized problem and thus one has a family of global solutions $\{u^\delta\,|\,\delta\in(0,1]\}$ and the considerations preceding Definition 5.1 yield the existence of a convergent subsequence and a limit with the stated properties. □

While Definition 5.1 is of the vanishing viscosity type, viscosity solution is not a viable concept for equation (5) due to the nonlocal nature of the nonlinearity and to the lack of a comparison principle. In fact it is not at all clear what equation is satisfied by the limiting function. The available a priori estimates also entail the existence of a Radon measure $\mu^0(t)$ a.e. in t such that

$$\triangle u^{\delta_j}(t)\stackrel{*}{\rightharpoonup}\mu^0$$

along a subsequence $(\delta_j)_{j\in\mathbb{N}}$. Clearly $\mu^0(t) = \triangle u^0$ since there is convergence of u^{δ_j} in the sense of distributions. Furthermore the simple inequality

$$\int_0^\infty \int_\Omega \frac{1}{\big(1+|(-\triangle)^{1-\varepsilon}u^\delta|^2\big)^2}|\triangle u^\delta|^2\,dxd\tau \leq \int_0^\infty \int_\Omega \frac{1}{1+|(-\triangle)^{1-\varepsilon}u^\delta|^2}|\triangle u^\delta|^2\,dxd\tau \leq c < \infty\,,$$

which is independent of δ, ensures the existence of a function $f \in \mathrm{L}_2\big([0,T]\times\Omega\big)$ such that

$$\frac{\triangle u^{\delta_j}}{1+|(-\triangle)^{1-\varepsilon}u^{\delta_j}|^2} \rightharpoonup f \in \mathrm{L}_2\big([0,T]\times\Omega\big)$$

so that $u^0_t = f$ in the limit. Again T is fixed but arbitrary. Unfortunately and to the best of our knowledge, it is not possible to establish a relation between f and u^0, that is, it is not clear that

$$f = \frac{1}{1+|(-\triangle)^{1-\varepsilon}u^0|^2}\triangle u^0\,.$$

Remark 4. Even if the latter relation could be established uniqueness of the solution of Definition 5.1 can not be obtained by standard means due to lack of knowledge concerning the location of the support of the singular part of $\triangle u^0$ of two potentially distinct solutions. Lack of uniqueness is known to occur for a kind of weak solution for the Perona-Malik equation (see [16, 17]).

Remark 5. The analysis performed in this paper does not exclude the possibility that solutions to smooth initial data do remain smooth forever, thus never developing a singularity. It would be very interesting to investigate this question as the numerical experiments cannot provide conclusive evidence in favor or against such an hypothesis.

Finally it should be remarked that a similar concept of weak solution can be introduced also for equation (4) but with the restriction that $n = 1$. Only on such an assumption it is in fact possible to derive L_1-bounds for u_x. This is because $v = u_x$ satisfies

$$v_t = \big(\frac{1}{1+|\partial_x^{1-\varepsilon}u|^2}v\big)_{xx}$$

an equation akin to that satisfied by $\triangle u$ for (5). This does not remain true for $n \geq 2$ where L_1-bounds are not known to be valid for the gradient of the solution.

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Received for publication September 2010.

E-mail address: gpatrick@math.uci.edu