J. Differential Equations 252 (2012) 3226–3244

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

A backward–forward regularization of the Perona–Malik equation

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ARTICLE INFO

Article history:
Received 18 February 2011
Revised 28 October 2011
Available online 6 December 2011

Keywords:
Nonlinear diffusion
Forward–backward diffusion
Well-posedness
Young measure solutions
Perona–Malik type equation
Global existence
Qualitative behavior

ABSTRACT

It is shown that the Perona–Malik equation (PME) admits a natural regularization by forward–backward diffusions possessing better analytical properties than PME itself. Well-posedness of the regularizing problem along with a complete understanding of its long time behavior can be obtained by resorting to weak Young measure valued solutions in the spirit of Kinderlehrer and Pedregal (1992) [1] and Demoulini (1996) [2]. Solutions are unique (to an extent to be specified) but can exhibit "micro-oscillations" (in the sense of minimizing sequences and in the spirit of material science) between "preferred" gradient states. In the limit of vanishing regularization, the preferred gradients have size 0 or ∞ thus explaining the well-known phenomenon of staircasing. The theoretical results do completely confirm and/or predict numerical observations concerning the generic behavior of solutions.

1. Introduction

Ever since it was proposed by Perona and Malik in 1990 [3], the so-called Perona–Malik equation (PME)

$$u_t = \nabla \cdot \left(\frac{1}{1+|\nabla u|^2} \nabla u \right) =: \nabla \cdot \left(a\left(|\nabla u|^2\right) \nabla u \right) \tag{1.1}$$

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.10.022

has attracted the interest of the mathematical community. It is an example of a nonlinear forward–backward heat equation which is not (directly) amenable to variational techniques. It is in fact the formal gradient flow generated by the convex-concave energy functional

$$E(u) = \frac{1}{2}\int_\Omega \log\bigl(1+|\nabla u|^2\bigr)\,dx = \int_\Omega \varphi\bigl(|\nabla u|\bigr)\,dx. \tag{1.2}$$

The arguably most natural variational approach to understand and solve this problem does not seem viable as the relaxation of the energy functional (1.2) vanishes everywhere; indeed $\widetilde{\varphi} \equiv 0$ for the convexification $\widetilde{\varphi}$ of φ. In an effort to overcome this difficulty Zhang and co-authors [4] and [5] introduced an ad-hoc concept of weak Young measure valued solution for (1.1) in one space dimension by rewriting the equation as a first order system. They proved existence of infinitely many such solutions (non-uniqueness) and obtained instability results. The class of solutions they consider exclude classical solutions. In this regard we observe that (1.1) does admit global in time classical solutions in the subcritical region where $|\nabla u| < 1$ which, as it turns out [6], is invariant under the evolution. In other words subcritical initial data yield classical solutions which remain subcritical for all times and converge to trivial (constant) steady-states. Such solutions cannot be captured by variational relaxation techniques which, as pointed out above, deliver a vanishing convexification and no evolution whatsoever for any initial datum. More recently, Smarrazzo and Tesei [7–9] obtained some general, albeit one-dimensional, results for forward–backward equations of PME type (not including PME) in the vanishing limit of a degenerate pseudo-parabolic regularization à la [10]. They use the concept of Young measure valued solution (different from the above one employed by Zhang), follow some ideas and techniques of [11–13] which rely on entropy inequalities. Many other regularization approaches were previously proposed over the years, starting with the early [14] up to the more recent [15–17], and through [18–25] to mention just a few. Other authors have obtained results concerning certain classes of classical solutions and/or their qualitative properties starting with [26], where ill-posedness in the one-dimensional setting is proved, and followed by [6,27–30]. Ghisi and Gobbino recently constructed an interesting family of global classical solutions of PME to transcritical initial conditions in two space dimensions. Finally there exist attempts at obtaining insights concerning the behavior of solutions to PME by semi-discretizations in space [31–33]. Many of the efforts mentioned were motivated by the desire to reconcile empirical, numerical observations about the behavior of solutions to PME with its mathematical properties so as to provide satisfactory theoretical explanations for them.

Here another, and arguably quite natural, approach is proposed by which the equation is very mildly regularized by a family of better natured forward–backward equations and which works in any space dimension. This novel regularization is even milder than that proposed in [15] but has the advantage of having a variational structure. The latter allows for an analysis based on general results outlined in [1] and obtained by Demoulini in [2]. The regularization simply reads

$$u_t = \nabla\cdot\left(\left[\frac{1}{1+|\nabla u|^2}+\delta\right]\nabla u\right) =: \nabla\cdot\bigl(a_\delta\bigl(|\nabla u|^2\bigr)\nabla u\bigr) \tag{1.3}$$

$$=: \nabla\cdot\bigl(q_\delta(\nabla u)\bigr). \tag{1.4}$$

While $0 < \delta \leqslant a_\delta \leqslant 1+\delta$ holds for the regularized problem (1.3), which makes the problem uniformly parabolic, it remains of forward–backward type for $\delta < 1/8$ and, thus, in the small δ regime of interest. The energy functional (1.2) is only mildly concave so that any $\delta > 0$ is enough to make

$$E_\delta(u) = \frac{1}{2}\int_\Omega \bigl\{\log\bigl(1+|\nabla u|^2\bigr)+\delta|\nabla u|^2\bigr\}\,dx =: \int_\Omega \varphi_\delta(\nabla u)\,dx \tag{1.5}$$

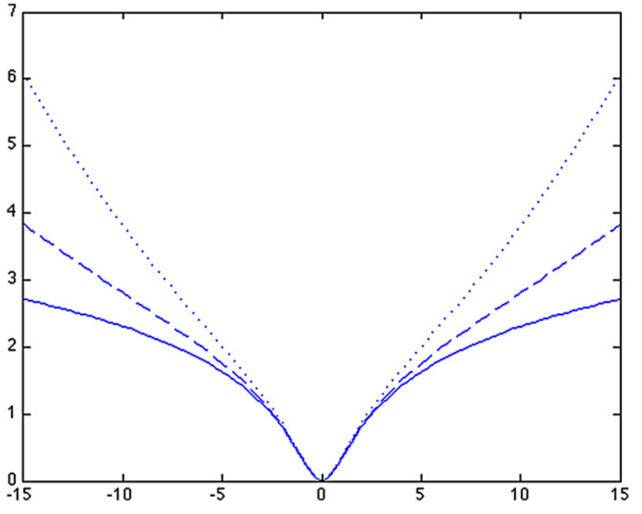

Fig. 1. The potential φ_δ for $\delta = 0, 0.01, 0.03$ (solid, dashed, and dotted line, respectively).

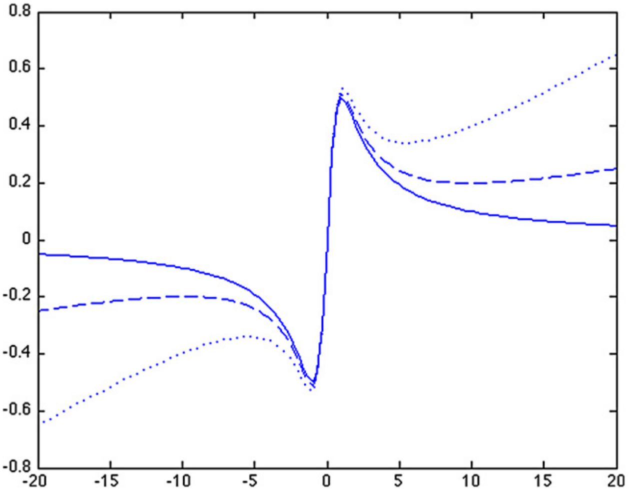

Fig. 2. The driving function $a_\delta(s^2)s$ for $\delta = 0, 0.01, 0.03$ (solid, dashed, and dotted line, respectively).

eventually convex. See Fig. 1. The original convex-concave energy E (1.2) is turned into the convex-concave-convex energy E_δ. To be more precise (1.3) can be rewritten as

$$\begin{aligned} u_t &= a_\delta\big(|\nabla u|^2\big)\Delta u + 2a'_\delta\big(|\nabla u|^2\big)\nabla u^T D^2 u \nabla u \\ &= a_\delta\big(|\nabla u|^2\big)\partial_{\tau\tau} u + \big\{a_\delta\big(|\nabla u|^2\big) + 2a'_\delta\big(|\nabla u|^2\big)|\nabla u|^2\big\}\partial_{\nu\nu} u \end{aligned} \tag{1.6}$$

by means of local coordinates in tangential τ and normal direction ν to the level sets of u. Depending on the size of $|\nabla u|$ diffusion in normal direction can still be backward. It is now, however, forward for $|\nabla u| \lessapprox 1$ and for $|\nabla u| \gtrapprox \frac{1}{\sqrt{\delta}}$ for $\delta \approx 0$. This behavior can be read off Fig. 2. The regions where the function $a_\delta(s^2)s$ is increasing and decreasing correspond to those where the equation is forward and

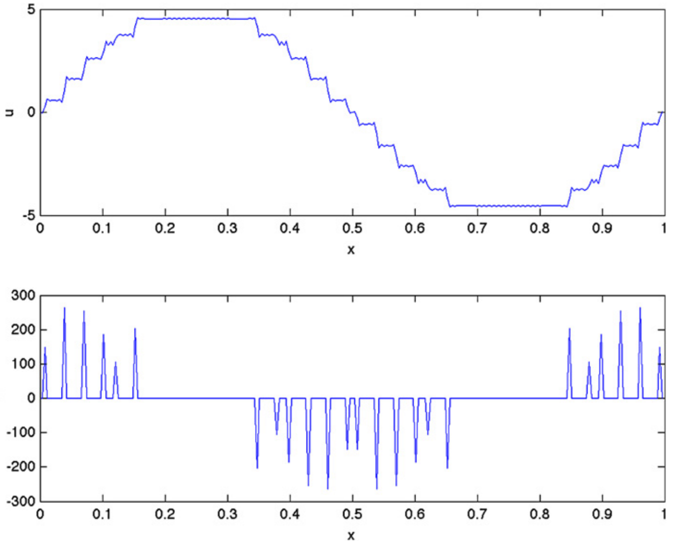

Fig. 3. The staircasing effect typically observed for the PME equation. The graphs of $u(t,\cdot)$ and $u_x(t,\cdot)$ are plotted at time $t=2$.

backward, respectively. Thus the backward regime of (1.3) is effectively confined to a finite interval which grows in size as the regularizing parameter vanishes. If the equation is rewritten in the form

$$u_t = \nabla \cdot \big(q_\delta(\nabla u)\big),$$

then the function q_δ has, for $\delta > 0$, the better understood and better behaved form of a cubic nonlinearity quite frequently encountered in the theory of phase transitions, at least locally around the backward regime, and has linear behavior at infinity.

It follows that unique solvability (to be made more precise later) of (1.3) can be obtained by using Young measures in a very natural way. Numerical experiments show that the presence of $\delta > 0$ effectively introduces a bound on the maximal size of gradients of solutions thus ensuring that a solution becomes instantaneously Lipschitz continuous even for initial data which are merely $\mathrm{L}_\infty(\Omega)$. The latter is the natural choice for the space of initial data in view of the applications of the equation to image processing and biology [3,34,35]. It follows that the well-known staircasing phenomenon observed for PME is replaced by a rather milder "jump-less" staircasing whereby a solution starts oscillating in a transiently failing attempt of fleeing the unstable regime by "randomly" choosing between small and large gradients in the forward regime. This gives rise to the Young measure component of the solution and is precisely the regime in which staircasing ($\delta = 0$) is replaced by "micro-structured ramping" ($0 < \delta < 1/8$), that is, the tendency of solutions to exhibit flat and bounded growth zones in a locally piecewise fashion. From the theoretical point of view, the uniform parabolicity of (1.3) entails at least the Hölder continuity of any bounded solution (see later for a more precise statement). The rather intriguing transition from rough to smooth staircasing is depicted in Figs. 3 and 4. It appears clearly that, in stark contrast with PME, its regularization (1.3) is much smoother in that its maximal gradients are confined to a bounded interval. The Young measure valued nature of the gradients is also apparent in the plotted time slice of the evolution.

In the limit as δ tends to zero, the solution is allowed to take on larger and larger gradients until, in the limit, jumps do appear and the solution looses its continuity. The maximal size of gradients grows in a quantifiable and controlled fashion, thus making it possible to obtain a limit for a rescaled version. In this regard it should be observed that the variational approach which allows to obtain a solution for (1.3) breaks down in the limit. This is due to the fact that the affine part of $\widetilde{\varphi}_\delta$ grows to cover the whole domain of definition as its "slope" decreases to zero ($\widetilde{\varphi} \equiv 0$). For such a situation the construction would deliver a solution which is frozen in the initial condition. Thus it is

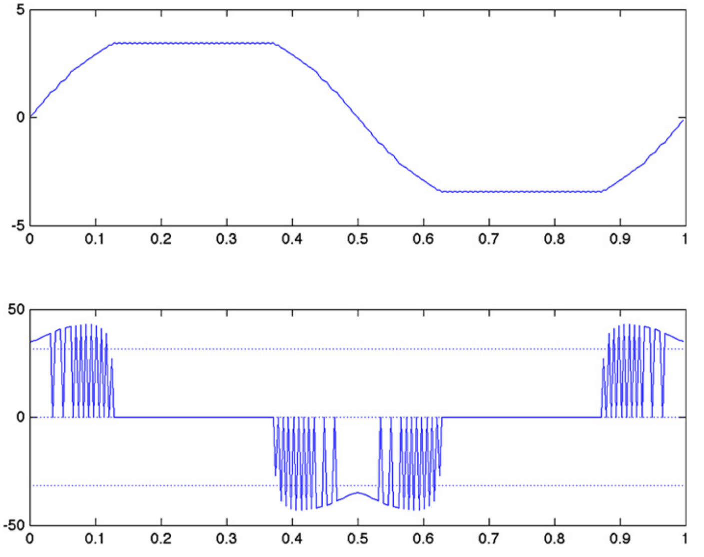

Fig. 4. The "gentler" staircasing effect observed for equation (1.3) for $\delta = 0.001$. The dotted lines correspond to approximated gradient values at which the transition between forward and backward regime occur. The graphs of $u(t,\cdot)$ and $u_x(t,\cdot)$ are plotted at time $t = 2$ of the evolution.

seen again that the only way to produce a meaningful limit would involve a rescaling of time in the process. After receiving a preliminary version of this paper, Colombo and Gobbino were able to use Gamma convergence techniques to carry out this very suggestion. Interestingly they found that carefully rescaled families of solutions u^δ do possess a limit for vanishing δ which does evolve according to the well-known Total Variation flow. It is referred to their paper [36] for the details.

It is worthwhile mentioning that the properties of regularization (1.3) have important consequences in applications to image processing. The latter will be considered in another more applied paper.

While the numerical observations made for the regularized problem cannot be all proven in their strongest form, they can certainly all be justified by the theoretical results presented here. They include global existence of weak Young measure valued solutions in any space dimension which do eventually converge to trivial steady-states and which admit Young measure representations of their gradients the support of which concentrate in the origin and at infinity in the vanishing regularization parameter limit. Young measures are well suited for capturing the oscillations in the solutions' gradient between preferred states which give rise to "micro-structured" gradients.

For the sake of completeness it should be observed that, while the class of solutions constructed in this paper do indeed satisfactorily predict and reflect the behavior observed in numerical simulations, it does exclude certain classical solutions of the original problem. This is due to the fact that the relaxation procedure employed here does modify the original energy in its forward (convex) region as well.

The paper is organized as follows. In the next section the Young measure valued approach to the solvability of certain forward–backward equations is presented. In Section 3 the results are applied to the regularized problem (1.3). The results of numerical experiments are shown all along in order to motivate and/or underscore the theoretical findings.

2. Young measures and forward–backward diffusions

The use of non-convex functionals in the calculus of variations for applications in material science has been discussed by Chipot and Kinderlehrer in [37] where they introduced a crucial stability criterion. Later Kinderlehrer and Pedregal [1] found sufficient conditions for the validity of the criterion in terms of growth properties of the potential φ, thus ensuring the availability of important Young measure representations. They also outlined applications of various nature including one to the resolution of evolution equations. Demoulini [2] finally obtained comprehensive results about existence, uniqueness, and qualitative properties of solutions to gradient systems satisfying these conditions in the case of homogeneous Dirichlet boundary conditions.

A description of the method is given here in which the results of Demoulini are translated from the Dirichlet into the periodic and Neumann settings considered in this paper. Given a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geqslant 1$) with piecewise smooth boundary $\partial\Omega$, of interest is the divergence form evolutionary equation

$$\begin{cases} u_t = \nabla \cdot \big(q(\nabla u)\big), & \text{in } Q_\infty := \mathbb{R}^+ \times \Omega, \\ q(\nabla u) \cdot \nu = 0, & \text{on } \partial Q_\infty := \mathbb{R}^+ \times \partial\Omega, \\ u(0, \cdot) = u_0, & \text{in } \Omega, \end{cases} \tag{2.1}$$

where the homogeneous Neumann conditions can be replaced by a periodicity assumption provided the domain is set to be a unit box. The structural assumption that

$$q = \nabla\varphi \quad \text{for } \varphi \in \mathrm{C}^1\big(\mathbb{R}^N\big)$$

is made along with the requirement that

$$\big(c|\xi|^2 - 1\big)^+ \leqslant \varphi(\xi) \leqslant C|\xi|^2 + 1, \quad \xi \in \mathbb{R}^N \quad \text{and} \tag{2.2}$$

$$0 \leqslant q(\xi) \cdot \xi, \qquad \big|q(\xi)\big| \leqslant C|\xi|, \quad \xi \in \mathbb{R}^N, \tag{2.3}$$

for some $0 < c \leqslant C < \infty$. Given a fixed but arbitrary $u_0 \in \mathrm{H}^1(\Omega)$, the space $\mathrm{H}^1_{u_0}(\Omega)$ will denote the affine space of $\mathrm{H}^1(\Omega)$-functions satisfying

$$\int_\Omega u\,dx = \int_\Omega u_0\,dx$$

or its subspace of periodic functions satisfying the same condition depending on the choice of boundary condition. It is also recalled that a *Young measure* on $\Omega \times \mathbb{R}^N$ is a positive measure μ such that

$$\mu\big(A \times \mathbb{R}^N\big) = \lambda_N(A), \quad A \in \mathcal{B}\big(\mathbb{R}^N\big),$$

where $\mathcal{B}(\mathbb{R}^N)$ is the collection of Borel sets of $\mathbb{R}^N$ and λ_N is the N-dimensional Lebesgue measure. It is possible to associate a Young measure to any given measurable function $z : \Omega \to \mathbb{R}^N$ by setting

$$\mu_z = (\delta_{z(x)})_{x\in\Omega}.$$

The *disintegration* $(\mu_x)_{x\in\Omega}$ of a Young measure μ (which does always exist, cf. [38]) is such that for any $f \in \mathrm{C}(\overline{\Omega} \times \mathbb{R}^N)$ one has

$$\langle \mu, f\rangle = \int_\Omega \int_{\mathbb{R}^N} f(x,\xi)\,d\mu_x(\xi)\,dx,$$

and thus, in particular,

$$\langle \mu_z, f \rangle = \int_\Omega f\bigl(x, z(x)\bigr)\, dx.$$

Finally it is said that μ is generated by a sequence of functions $(z_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^1(\Omega)^N$ whenever

$$\int_\Omega f\bigl(x, z_n(x)\bigr)\, dx = \langle \mu_{z_n}, f \rangle \to \langle \mu, f \rangle, \quad f \in \mathrm{C}_c\bigl(\Omega \times \mathbb{R}^N\bigr).$$

The point is here of course that μ does not need to be, and, in general is not, a Young measure associated to any function. This allows the concept to capture the limiting behavior of certain increasingly oscillating sequences of functions (cf. [39]). Notice that the spaces $\mathrm{H}^1_{loc}(Q_\infty)$ and $\mathrm{H}^1_{0,loc}(Q_\infty)$ are defined through

$$\mathrm{H}^1_{loc}(Q_\infty) = \bigl\{u : Q_\infty \to \mathbb{R} \bigm| u \in \mathrm{H}^1(Q_T) \text{ for } T > 0\bigr\},$$

and

$$H^1_{0,loc}(Q_\infty) = \bigl\{u : Q_\infty \to \mathbb{R} \bigm| u \in \mathrm{H}^1(Q_T) \text{ for } T > 0 \text{ and } u(\cdot, t) \in \mathrm{H}^1_0(\Omega) \text{ a.e. in } Q_\infty\bigr\},$$

respectively.

Definition 1. A *weak Young measure valued solution* of (2.1) is a pair (u, ν) where

$$u \in \mathrm{H}^1_{loc}(Q_\infty) \cap \mathrm{L}^\infty\bigl(\mathbb{R}^+, \mathrm{H}^1(\Omega)\bigr)$$

and $\nu = (\nu_{x,t})_{(x,t)\in Q_\infty}$ is a Young measure generated by a sequence of spatial gradients (as will be explained below) such that

$$\int_{Q_T} \bigl\{u_t \psi + \langle \nu, q \rangle \nabla \psi\bigr\}\, dx\, dt = 0, \quad \psi \in \mathrm{H}^1(Q_T),\ T > 0,$$

and such that they are connected through

$$\nabla u(t, x) = \int_{\mathbb{R}^N} \xi \, d\nu_{x,t}(\xi) \quad \text{a.e. in } Q_\infty.$$

Remark 2. By inserting the constant test function $\psi = 1_\Omega$ with value 1 in the above weak equation, it readily follows that the mean of any solution is a conserved quantity

$$\frac{d}{dt} \int_\Omega u\, dx = -\int_\Omega \langle \nu, q \rangle 0\, dx = 0.$$

From now on, it will therefore be required for a solution to satisfy $u(t) \in \mathrm{H}^1_{u_0}(\Omega)$ a.e. in $t \geqslant 0$ and that the weak formulation be satisfied for all test functions in the subspace $\mathrm{H}^1_0(Q_\infty)$.

Following [1] and [2], a solution of (2.1) is obtained by semi-discretization in time and passage to the limit. Let $h > 0$ and let $u^{j,h}$ $(j \geqslant 1)$ be given as the minimizer of the functional

$$\widetilde{E}^h_{u^{h,j-1}}(v) := \int_\Omega \widetilde{\varphi}(\nabla v)\,dx + \frac{1}{2h}\int_\Omega \big(v - u^{h,j-1}\big)^2\,dx, \quad v \in \mathrm{H}^1(\Omega),$$

over $v \in \mathrm{H}^1(\Omega)$ where $u^{h,0} := u_0$. Since φ is not necessarily convex, and following relaxation theory, φ is replaced by its convexification $\widetilde{\varphi}$. The energy functional $\widetilde{E}_h$ has the same infimum as

$$E^h_{u^{h,j-1}}(v) := \int_\Omega \varphi(\nabla v)\,dx + \frac{1}{2h}\int_\Omega \big(v - u^{h,j-1}\big)^2\,dx, \quad v \in \mathrm{H}^1(\Omega).$$

It is readily seen that

$$\widetilde{E}^h_{u_0}\bigg(v - \fint_\Omega v\,dx + \fint_\Omega u_0\,dx\bigg) \leqslant \widetilde{E}^h_{u_0}(v), \quad v \in \mathrm{H}^1(\Omega),$$

and that

$$E^h_{u_0}\bigg(v - \fint_\Omega v\,dx + \fint_\Omega u_0\,dx\bigg) \leqslant E^h_{u_0}(v), \quad v \in \mathrm{H}^1(\Omega).$$

It follows that minimizers and minimizing sequences can all be assumed to lie in $\mathrm{H}^1_{u_0}(\Omega)$ without loss of generality, in keeping with Remark 2. By defining

$$u^h : \mathbb{R}^+ \to \mathrm{H}^1_{u_0}(\Omega)$$

on the whole positive axis by linear interpolation on the intervals $[(j-1)h, jh)$ using the sequence $(u^{h,j})_{j\in\mathbb{N}}$, and setting ν^h to be piecewise constant on the same intervals using the Young measure associated with the sequence of gradients $\nabla u^{h,j}_n$ of a minimizing sequence $(u^{h,j}_n)_{n\in\mathbb{N}}$ for the corresponding energy functional, an approximating sequence is obtained for the solution of the continuous time problem which satisfies the equation a.e. in time. A priori estimates are available as a consequence of the dissipativity of the equation reflected in

$$\int_\Omega \varphi\big(\nabla u^{h,j}\big)\,dx + \frac{1}{2h}\int_\Omega \big(u^{h,j} - u^{h,j-1}\big)^2\,dx \leqslant \int_\Omega \varphi\big(\nabla u^{h,j-1}\big)\,dx, \quad j \geqslant 1,$$

and allow for the construction of a limit. One eventually obtains a solution of the original problem as summarized in the following theorem along with some qualitative properties which are a nice consequence of the construction. While an almost complete proof of the result will be given here, the interested reader is referred to [2] for further details of the arguments used in it modulo, of course, the necessary changes caused by the different functional setting.

Theorem 3. *Let the assumptions* (2.2)–(2.3) *be satisfied and take* $u_0 \in \mathrm{H}^1(\Omega)$. *Then problem* (2.1) *possesses a unique*[1] *weak Young measure valued solution* (u, ν). *Uniqueness only pertains to the function* u *while the*

[1] See Remark 4 following the proof of the theorem.

(gradient) Young measure $\nu = (\nu_{x,t})_{x,t \in Q_\infty}$ is in general not unique. It is also the solution of the relaxed problem

$$u_t = \nabla \cdot p(\nabla u),$$

where $p = \nabla \widetilde{\varphi}$. One also has that

$$u_t \in \mathrm{L}^2(Q_\infty),$$

that

$$\operatorname{supp}(\nu_{x,t}) \subset [\varphi = \widetilde{\varphi}] \cap A(\widetilde{\varphi}) \quad \text{a.e. in } Q_\infty,$$

and that

$$\nabla u(x,t) = \int_{\mathbb{R}^N} \xi \, d\nu_{x,t}(\xi) \quad \text{a.e. in } Q_\infty.$$

Here $A(\widetilde{\varphi})$ denotes the collection of all subsets on which $\widetilde{\varphi}$ behaves in an affine manner. Any solution eventually converges to a Young measure valued solution (u^∞, ν^∞) of the corresponding stationary problem in the sense that

$$\begin{aligned} u(t,\cdot) &\rightharpoonup u^\infty \quad \text{in } \mathrm{H}^1_{u_0}(\Omega) \text{ as } t \to \infty, \\ \nabla u^\infty &= \int_{\mathbb{R}^N} \xi \, d\nu^\infty(\xi) \quad \text{a.e. in } \Omega, \end{aligned} \tag{2.4}$$

and it holds that

$$\operatorname{supp}\bigl(\nu^\infty\bigr) \subset \bigl[\xi \cdot q(\xi) = 0\bigr] \cap [\varphi = \widetilde{\varphi} = 0].$$

The comparison principle holds, that is,

$$-\bigl\|(u_0 - v_0)^-\bigr\|_\infty \leqslant u(t,x) - v(t,x) \leqslant \bigl\|(u_0 - v_0)^+\bigr\|_\infty \quad \text{a.e. in } Q_\infty$$

is valid for any pair (u,ν), (v,μ) of Young measure valued solutions of (2.1) *with initial data u_0 and v_0, respectively.*

Proof. Let $h > 0$ and consider the following equation

$$\frac{1}{h}(u - w) = \nabla \cdot p(\nabla u),$$

which can clearly be viewed as a single time step discretization of the gradient flow

$$u_t = \nabla \cdot p(\nabla u), \quad t > 0, \qquad u(0) = w,$$

corresponding to the convexified problem. The idea is to construct a sequence by recursively solving this equation for u given w starting with $w = u_0$ and subsequently with $w = u^{h,j}$ where $u^{h,j}$ is the solution computed last. Since, for any given $u^{h,j-1} \in \mathrm{H}^1_{u_0}(\Omega)$, the equations are the Euler–Lagrange equations for the strictly convex variational problem

$$\operatorname{argmin} \int_\Omega \left\{ \widetilde{\varphi}(\nabla v) + \frac{1}{2h} \left| v - u^{h,j-1} \right|^2 \right\} dx = \operatorname{argmin} \widetilde{E}^h_{u^{h,j-1}}(v), \quad v \in \mathrm{H}^1(\Omega),$$

this amounts to recursive minimization of the functionals $\widetilde{E}^h_{u^{h,j-1}}$. Now a sequence $(u^{h,j,k})_{k\in\mathbb{N}}$ can always be chosen such that it approximates the minimum $u^{h,j}$ of $\widetilde{E}^h_{u^{h,j-1}}$ and such that it simultaneously is a minimizing sequence for

$$E^h_{u^{h,j-1}}(v) = \int_\Omega \left\{ \varphi(\nabla v) + \frac{1}{2h} \left| v - u^{h,j-1} \right|^2 \right\} dx, \quad v \in \mathrm{H}^1(\Omega),$$

since one function is the relaxation of the other. The growth assumptions on φ entail that

$$\left\| u^{h,j,k} \right\|_{\mathrm{H}^1(\Omega)} \leqslant c < \infty, \quad k \in \mathbb{N}.$$

There exists therefore $u^{h,j} \in \mathrm{H}^1_{u_0}(\Omega)$ such that

$$\begin{aligned} u^{h,j,k} &\rightharpoonup u^{h,j} \quad \text{in } \mathrm{H}^1_{u_0}(\Omega), \\ u^{h,j,k} &\to u^{h,j} \quad \text{in } \mathrm{L}^2(\Omega), \end{aligned}$$

as $k \to \infty$, by compactness of the embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$. Clearly

$$\begin{aligned} \inf E^h_{u^{h,j-1}} &= \lim_{k\to\infty} E^h_{u^{h,j-1}}\big(u^{h,j,k}\big) = \lim_{k\to\infty} \widetilde{E}^h_{u^{h,j-1}}\big(u^{h,j,k}\big) \\ &= \min \widetilde{E}^h_{u^{h,j-1}} = \widetilde{E}^h_{u^{h,j-1}}\big(u^{h,j}\big). \end{aligned} \tag{2.5}$$

Again by means of the growth assumptions on φ (and thus on $\widetilde{\varphi}$) and using the main results of [1] (also restated in the introduction of [2]) it follows that

$$\widetilde{\varphi}\big(\nabla u^{h,j,k}\big) \rightharpoonup \widetilde{\varphi}\big(\nabla u^{h,j}\big) \quad \text{in } \mathrm{L}^1(\Omega).$$

Denote by $\nu^{h,j}$ the Young measures generated by the sequences $(\nabla u^{h,j,k})_{k\in\mathbb{N}}$. By [2, Theorem 2.3] it is an $\mathrm{H}^1(\Omega)$-gradient Young measure and, once more thanks to the growth assumptions on φ, one has that $\varphi(\nabla u^{h,j,k})$ is weakly convergent in $\mathrm{L}^1(\Omega)$ and, using (2.5), that

$$\int_\Omega \widetilde{\varphi}\big(\nabla u^{h,j}\big)\, dx = \int_\Omega \big\langle \nu^{h,j}, \widetilde{\varphi} \big\rangle\, dx = \int_\Omega \big\langle \nu^{h,j}, \varphi \big\rangle\, dx.$$

It should be pointed out that [1, Theorem 1.3] plays a crucial role here. Since $\widetilde{\varphi} \leqslant \varphi$ by definition of convexification, it follows that

$$\begin{gathered} \operatorname{supp}\big(\nu^{h,j}\big) \subset [\varphi = \widetilde{\varphi}] \quad \text{a.e. in } \Omega, \\ \big\langle \nu^{h,j}, \varphi \big\rangle = \big\langle \nu^{h,j}, \widetilde{\varphi} \big\rangle = \widetilde{\varphi}\big(\nabla u^{h,j}\big) \quad \text{a.e. in } \Omega, \\ \nabla u^{h,j} = \big\langle \nu^{h,j}, \mathrm{id} \big\rangle \quad \text{a.e. in } \Omega. \end{gathered}$$

Using $[\nabla\varphi = \nabla\widetilde{\varphi}] \supset [\varphi = \widetilde{\varphi}]$ one also infers that

$$\langle \nu^{h,j}, q\rangle = \langle \nu^{h,j}, p\rangle \quad \text{a.e. in } \Omega.$$

Jensen's inequality yields

$$\widetilde{\varphi}\big(\langle \nu^{h,j}, \mathrm{id}\rangle\big) \leqslant \langle \nu^{h,j}, \widetilde{\varphi}\rangle = \widetilde{\varphi}\big(\nabla u^{h,j}\big) = \widetilde{\varphi}\big(\langle \nu^{h,j}, \mathrm{id}\rangle\big) \quad \text{a.e. in } \Omega, \tag{2.6}$$

which, in turn, entails that $\mathrm{supp}(\nu^{h,j})$ lies in a connected region where $\nabla\widetilde{\varphi}$ is constant because equality in Jensen's inequality only holds in the affine case.

It is plain that the minimizer $u^{h,j}$ satisfies the corresponding Euler–Lagrange equation, i.e., that

$$\nabla \cdot p\big(\nabla u^{h,j}\big) = \frac{1}{h}\big(u^{h,j} - u^{h,j-1}\big) \quad \text{in } \mathrm{H}^{-1}(\Omega) = \mathrm{H}^1_0(\Omega)',$$

or, equivalently that

$$\big\langle p\big(\nabla u^{h,j}\big), \nabla\psi\big\rangle = \frac{1}{h}\big\langle u^{h,j} - u^{h,j-1}, \psi\big\rangle, \quad \psi \in \mathrm{H}^1_0(\Omega). \tag{2.7}$$

Now, using an argument of [1,37], consider

$$\widetilde{\varphi}\big(\xi + \varepsilon\nabla\psi(x)\big) \leqslant C\big(1 + |\xi|^2\big), \quad \xi \in \mathbb{R}^N,$$

for $\psi \in \mathrm{C}^1_0(\overline{\Omega}) := \{\psi \in \mathrm{C}^1(\Omega) \mid \int_\Omega \psi = 0\}$ such that $|\nabla\psi| \in \mathrm{L}^\infty(\Omega)$ and for $\varepsilon \in [-1, 1]$. Then

$$\begin{aligned}
&\int_\Omega \bigg[\big\langle \nu^{h,j}, \varphi\big\rangle + \frac{1}{2h}\big(u^{h,j} - u^{h,j-1}\big)^2\bigg]\,dx \\
&\quad = \int_\Omega \bigg[\big\langle \nu^{h,j}, \widetilde{\varphi}\big\rangle + \frac{1}{2h}\big(u^{h,j} - u^{h,j-1}\big)^2\bigg]\,dx \\
&\quad \leqslant \lim_{k\to\infty} \int_\Omega \bigg[\big\langle \nu^{h,j}, \widetilde{\varphi}\big(\nabla u^{h,j,k} + \varepsilon\nabla\psi\big)\big\rangle + \frac{1}{2h}\big(u^{h,j} + \varepsilon\psi - u^{h,j-1}\big)^2\bigg]\,dx \\
&\quad = \int_\Omega \bigg[\big\langle \nu^{h,j}, \widetilde{\varphi}(\cdot + \varepsilon\nabla\psi)\big\rangle + \frac{1}{2h}\big(u^{h,j} + \varepsilon\psi - u^{h,j-1}\big)^2\bigg]\,dx \\
&\quad \leqslant \int_\Omega \bigg[\big\langle \nu^{h,j}, \varphi(\cdot + \varepsilon\nabla\psi)\big\rangle + \frac{1}{2h}\big(u^{h,j} + \varepsilon\psi - u^{h,j-1}\big)^2\bigg]\,dx
\end{aligned}$$

and differentiating in $\varepsilon = 0$ yields

$$\begin{aligned}
&-\nabla \cdot \big\langle \nu^{h,j}, q\big\rangle + \frac{1}{h}\big(u^{h,j} - u^{h,j-1}\big) \\
&\quad = -\nabla \cdot \big\langle \nu^{h,j}, p\big\rangle + \frac{1}{h}\big(u^{h,j} - u^{h,j-1}\big) = 0 \quad \text{in } \mathrm{H}^{-1}(\Omega).
\end{aligned} \tag{2.8}$$

To conclude the latter it is necessary to show that any $\psi \in \mathrm{H}_0^1(\Omega)$ can be approximated by $\psi_\varepsilon \in \mathrm{C}_0^1(\overline{\Omega})$ with $\|\nabla\psi_\varepsilon\|_\infty \leqslant C_\varepsilon < \infty$. This, however, follows from the boundedness of the Helmholtz projection

$$P : \mathrm{L}^2(\Omega)^N \to \mathrm{L}^2(\Omega)^N, \quad u \mapsto Pu$$

in $\mathrm{L}_2(\Omega)$ onto divergence free vector fields. Consider, in fact, a cut-off function $\chi_\varepsilon \in \mathrm{C}_c^\infty(\Omega)$ and a mollifier $\varphi_\varepsilon \in \mathrm{C}_c^\infty(\Omega)$. Then

$$(1-P)\varphi_\varepsilon * (\nabla\psi \cdot \chi_\varepsilon) =: \nabla\psi_\varepsilon \in \mathrm{C}^\infty(\overline{\Omega})$$

for some $\psi_\varepsilon \in \mathrm{C}^\infty(\overline{\Omega})$ with $\int_\Omega \psi_\varepsilon\, dx = 0$. Since

$$\varphi_\varepsilon * (\nabla\psi \cdot \chi_\varepsilon) \to \nabla\psi \quad \text{in } \mathrm{L}^2(\Omega),$$

one also has that

$$\nabla\psi_\varepsilon \to (1-P)\nabla\psi = \nabla\psi \quad \text{in } \mathrm{L}^2(\Omega).$$

By Poincaré's inequality $\|\psi - \psi_\varepsilon\|_2 \leqslant C\|\nabla\psi - \nabla\psi_\varepsilon\|_2$ and the claim follows.

Define now

$$\begin{aligned} u^h(x,t) &= u^{h,j}(x) + (t-jh)\frac{1}{h}\big(u^{h,j+1} - u^{h,j}\big), \quad x \in \Omega, \\ w^h(x,t) &= u^{h,j}(x), \quad x \in \Omega, \\ \nu^h &= \big(\nu_x^{h,j}\big)_{x\in\Omega}, \end{aligned}$$

for $t \in [jh, (j+1)h)$, $j \geqslant 0$. Then

$$u^h, w^h \in \mathrm{L}^\infty\big([0,\infty), \mathrm{H}^1_{u_0}(\Omega)\big), \qquad \nu^h \in \mathrm{L}^1_{loc}\big(Q_\infty, \mathcal{E}_0^1\big) \cap \mathrm{L}^\infty\big(Q_\infty, \mathcal{M}\big(\mathbb{R}^N\big)\big)$$

uniformly in $h > 0$ (in the sense that a bound exists in the corresponding norm which is independent of $h > 0$). Here the notation

$$\mathcal{E}_0^1 := \left\{ q \in \mathrm{C}\big(\mathbb{R}^N\big) \;\Big|\; \lim_{|\xi|\to\infty} \frac{|q(\xi)|}{1+|\xi|} \text{ exists} \right\}$$

was used following [2]. Observe that

$$\big\langle \nu^h, q \big\rangle \in \mathrm{L}^\infty\big([0,\infty), \mathrm{L}^2(\Omega)\big),$$

and that

$$u_t^h = \frac{1}{h}\big(u^{h,j+1} - u^{h,j}\big) \quad \text{on } \big[jh, (j+1)h\big).$$

It follows that $\{u_t^h \mid h > 0\}$ is bounded in $\mathrm{L}^2(Q_\infty)$ since

$$\int_0^\infty \int_\Omega \big|u_t^h\big|^2 \,dx\,dt = \sum_{j\geqslant 1} \frac{1}{h} \int_\Omega \big(u^{h,j+1} - u^{h,j}\big)^2 \,dx \leqslant \int_\Omega \varphi(\nabla u_0)\,dx,$$

where the inequality holds by construction. Given $T > 0$, (2.7) and (2.8) yield

$$\int_0^T \int_\Omega \left\{ \langle \nu^h, q \rangle \cdot \nabla \psi + u_t^h \psi \right\} dx\, dt = 0, \quad \psi \in \mathrm{H}_0^1(Q_T),$$

by density of tensor products $\psi(x,t) = \psi_1(x)\psi_2(t)$ in $\mathrm{H}_0^1(Q_T)$. Clearly one has $\langle \nu^h, q \rangle = \langle \nu^h, p \rangle$ for $t > 0$ and a.e. in Ω as well as

$$\nabla \cdot \langle \nu^h, q \rangle = \nabla \cdot \langle \nu^h, p \rangle = \nabla \cdot p(\nabla u^h) \quad \text{for } t > 0 \text{ in } \mathrm{H}^{-1}(\Omega)$$

and in $\mathrm{H}^{-1}(Q_T)$ as seen above for any $T > 0$. The family $(\nu^h)_{h>0}$ is uniformly bounded in

$$\mathrm{L}^\infty\left([0,\infty), \mathrm{L}^\infty\left(\Omega, \left(\mathcal{E}_0^1\right)'\right)\right) = \mathrm{L}^1\left(Q_\infty, \mathcal{E}_0^1\right)'$$

and $(\langle \nu^h, q \rangle)_{h>0}$ is uniformly bounded in

$$\mathrm{L}^\infty\left([0,\infty), \mathrm{L}^2(\Omega)\right) = \mathrm{L}^1\left([0,\infty), \mathrm{L}^2(\Omega)\right)'.$$

Moreover u^h and w^h are uniformly bounded in

$$\mathrm{H}_{loc}^1(Q_\infty) \cap \mathrm{L}^\infty\left([0,\infty), \mathrm{H}_{u_0}^1(\Omega)\right).$$

By weak compactness a subsequence can be found, which is not relabeled, converging to (u, ν) where u is the common limit of u^h and w^h in $\mathrm{L}_{loc}^2(Q_\infty)$ (as shown at the end of [1]). It holds that

$$u \in \mathrm{H}_{loc}^1(Q_\infty) \cap \mathrm{L}^\infty\left([0,\infty), \mathrm{H}_{u_0}^1(\Omega)\right), \quad u_t \in \mathrm{L}_2(Q_\infty),$$

that, for $T > 0$,

$$\int_0^T \int_\Omega \left\{ \langle \nu, q \rangle \cdot \nabla \psi + u_t \psi \right\} dx\, dt = 0, \quad \psi \in \mathrm{H}_0^1(Q_T),$$

and that

$$\langle \nu, q \rangle = \langle \nu, p \rangle \quad \text{a.e. in } Q_\infty, \qquad \operatorname{supp}(\nu) \subset [\varphi = \widetilde{\varphi}] \cap A(\widetilde{\varphi}).$$

In the same manner as in [2, Corollary 3.2], ν can be shown to be a gradient generated Young measure associated to a sequence in $\mathrm{L}^2([0,\infty), \mathrm{H}_{u_0}^1(\Omega))$ by a diagonal argument relating it back to $(\nabla u^{h,j,k})_{k\in\mathbb{N}}$. It holds that

$$\nabla u(x,t) = \langle \nu_{x,t}, \mathrm{id} \rangle \quad \text{a.e. in } Q_\infty,$$

because $\nabla w^h \rightharpoonup \nabla u$ in $\mathrm{L}_{loc}^2(Q_\infty)$, $\nabla w^h = \langle \nu^h, \mathrm{id} \rangle$, and uniqueness of weak limits.

Uniqueness of the solution follows the argument described in [2] and relies on the following independence property

$$\langle \nu_{x,t}, q \cdot \mathrm{id} \rangle = \langle \nu_{x,t}, q \rangle \cdot \langle \nu_{x,t}, \mathrm{id} \rangle. \tag{2.9}$$

The comparison principle and the claim concerning the long time behavior can also be obtained in a similar manner as in [2]. □

Remark 4. It should be pointed out that the uniqueness result only applies to solutions which satisfy the independence condition (2.9), respect the relation between the original and the relaxed problem in the sense that

$$\langle \nu, q\rangle = \langle \nu, p\rangle$$

and for which $\nabla u(x,t) = \langle \nu_{x,t}, \mathrm{id}\rangle$.

3. The regularized equation

It is now possible to formulate the main result concerning the regularized problem (1.3). It is straightforward to verify that

$$\begin{aligned}
&0 < \delta \leqslant a_\delta(s) \leqslant 1+\delta, \quad s \geqslant 0,\\
&\frac{\delta}{2}|\xi|^2 \leqslant \varphi_\delta(\xi) \leqslant \left(1+\frac{\delta}{2}\right)|\xi|^2, \quad \xi \in \mathbb{R}^N,\\
&0 \leqslant q_\delta(\xi)\cdot\xi, \qquad \left|q_\delta(\xi)\right| \leqslant 2|\xi|, \quad \xi \in \mathbb{R}^N.
\end{aligned} \tag{3.1}$$

Theorem 5. *Let $\delta \in (0, 1/8]$ be given. Problem (1.3) possesses a unique global solution $u_\delta : \mathbb{R}^+ \to \mathrm{H}^1_{u_0}(\Omega)$ for any given $u_0 \in \mathrm{H}^1(\Omega)\cap \mathrm{L}^\infty(\Omega)$ such that there exists a parameterized Young measure $(\nu^\delta_{x,t})_{(x,t)\in Q_\infty}$ satisfying*

$$u_\delta \in \mathrm{L}^\infty(\Omega), \quad u_\delta \in \mathrm{C}^{\alpha,\alpha/2}(Q_\infty),\ u_{\delta t} \in \mathrm{L}^2(Q_\infty),$$

and

$$\int_\Omega u_{\delta t}\psi\,dx + \int_\Omega\int_{\mathbb{R}^N} q_\delta(\xi)\cdot\nabla\psi\,d\nu^\delta_{x,t}(\xi)\,dx = 0, \quad \psi \in \mathrm{H}^1_{0,loc}(Q_\infty).$$

Furthermore

$$\begin{aligned}
&\operatorname{supp}\left(\nu^\delta_{x,t}\right) \subset [\varphi_\delta = \widetilde{\varphi}_\delta] \subset \left[|\xi| \leqslant m_\delta\right] \cup \left[|\xi| \geqslant M_\delta\right] \quad \textit{and}\\
&\nabla u_\delta(x,t) = \int_{\mathbb{R}^N} \xi\,d\nu^\delta_{x,t}(\xi) \quad \textit{a.e. in } Q_\infty,
\end{aligned}$$

where $m_\delta \lesssim \sqrt{\delta}$ and $M_\delta \gtrsim \frac{1}{\sqrt{\delta}}$. For almost any (x,t), the support of $\nu_{x,t}$ is either a singleton or contained in a region where $\widetilde{\varphi}$ is constant (see Remark 7 below).

Furthermore

$$u_\delta(t,\cdot) \rightharpoonup \int_\Omega u_0(x)\,dx \quad \textit{in } \mathrm{H}^1(\Omega) \textit{ as } t \to \infty.$$

Proof. It is a consequence of (3.1) that Theorem 3 applies to the forward–backward equation (1.3). Thus existence of a weak Young measure valued solution $u : [0,\infty) \to \mathrm{H}^1_{u_0}(\Omega)$ follows. The comparison principle and the assumption that $u_0 \in \mathrm{L}^\infty(\Omega)$ entail that

$$\left\| u(t,\cdot)\right\|_\infty \leqslant \|u_0\|_\infty, \quad t \geqslant 0.$$

Hölder regularity of u follows from the fact that it is in fact a bounded weak solution of

$$u_t = \nabla \cdot p_\delta(\nabla u) \quad \text{for } t > 0 \text{ in } \Omega,$$

that $\delta|\xi|^2 \leqslant p_\delta(\xi)\cdot\xi \leqslant C|\xi|^2$, $\xi \in \mathbb{R}^N$, and from [40,41]. Notice that $p_\delta = \nabla\widetilde{\varphi}_\delta$ for the convexification $\widetilde{\varphi}_\delta$ of φ_δ. It therefore satisfies the claimed classical regularity for some positive α. Next using the result about the support of the Young measure contained in Theorem 3, computing the locations where convexity inversion occurs for φ_δ, and estimating the locations where φ_δ and $\widetilde{\varphi}_\delta$ touch, it is straightforward to verify the claim about the support of $\nu_{x,t}$. Next observe that

$$\left[\xi\cdot q_\delta(\xi) = 0\right] \cap [\varphi_\delta = \widetilde{\varphi}_\delta] = \{0\}$$

which, by Theorem 3, implies that $\nabla u^\infty \equiv 0$ for any potential equilibrium of the evolution. As the average is a conserved quantity of the evolution, a solution originating in u_0 must eventually converge to its average, as stated. The Young measure representation of the gradients follows from Theorem 3. □

Remark 6. It is easy to see that the convexification $\widetilde{\varphi}_\delta$ of φ_δ coincide in a neighborhood of $\xi = 0$ and of $|\xi| = \infty$. In the central region, which grows as $\delta \to 0$, one has that

$$\widetilde{\varphi}(\xi) = \alpha_\delta \frac{\xi}{|\xi|}, \quad \xi \in \left[m_\delta \leqslant |\xi| \leqslant M_\delta\right]$$

for some $0 < \alpha_\delta \to 0$ $(\delta \to 0)$ by rotational symmetry. This helps understand the Γ-convergence result of [36] to solutions of the Total Variation flow. Roughly speaking, rescaling is needed in order to prevent α_δ from vanishing in the limit but also erases all information concerning the time scale for which the onset of oscillations is observed.

Remark 7. Almost everywhere in Q_∞ one has either that $\operatorname{supp}\nu_{x,t}$ is a singleton, in which case

$$\nu_{x,t} = \delta_\xi \quad \text{for } \xi = \xi(x,t) \in [\varphi = \widetilde{\varphi}],$$

or is contained in a region where p_δ is constant (see the proof of Theorem 3). Due to the rotational symmetry of $\widetilde{\varphi}$ such regions are radial segments of the form

$$\left\{ r\frac{\xi_0}{|\xi_0|} \,\middle|\, r \in [m_\delta, M_\delta]\right\} \quad \text{for a } \xi_0 \in \mathbb{R}^N,$$

which, combined with the requirement that $\operatorname{supp}\nu \subset [\varphi = \widetilde{\varphi}]$, yields

$$\nu_{x,t} = \gamma\delta_{\lambda^\delta_{x,t}} + (1-\gamma)\delta_{\Lambda^\delta_{x,t}},$$

where $\lambda^\delta_{x,t} = m_\delta\frac{\xi_0}{|\xi_0|}$ and $\Lambda^\delta_{x,t} = M_\delta\frac{\xi_0}{|\xi_0|}$ for some $\xi_0 = \xi_0(x,t) \in \mathbb{R}^N$, and where $\gamma = \gamma(x,t) \in [0,1]$.

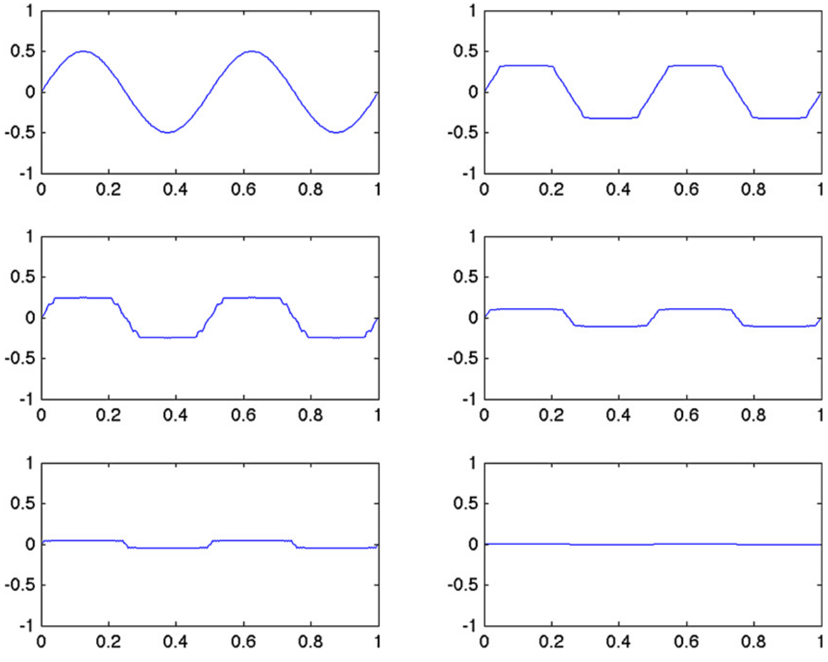

Fig. 5. Long time behavior of a solution to (1.3).

Remark 8. An interesting consequence of Theorem 5 is that any solution u^δ of (1.3) has an essentially bounded gradient in the backward region where oscillations may occur since

$$\left|\nabla u^\delta(x,t)\right| = \left|\left\langle \nu^\delta_{x,t}, \mathrm{id}\right\rangle\right| \leqslant \max_{\xi \in \mathrm{supp}(\nu_{x,t})} |\xi| \leqslant M_\delta < \infty,$$

because ν is a family of probability measures and M_δ is a bound for the set $[\varphi \neq \widetilde{\varphi}]$.

Remark 9. It is interesting to observe that solutions of (1.3) are still continuous, while the same cannot be said for all solutions of the limiting PME.

Remark 10. The concept of weak Young measure valued solution for (1.3) is rather natural in hindsight. It is in fact able to capture the oscillatory behavior of the gradient caused by the attempt of solutions to flee the backward regime either developing smaller or larger gradients (see Remark 7) in an alternating fashion. While this might be the generic behavior of global solutions of PME to transcritical initial data, interestingly there do exist global classical solutions which are initially transcritical. They clearly need to have a very special structure, but there are whole families of them as shown in [42].

Remark 11. The long time behavior of solutions, that is, the eventual convergence to a trivial steady-state for (1.3) as well as for (PME) has long been observed numerically. It is comforting to have a theoretical confirmation that this is not an artifact due to numerical diffusion but rather a feature of the equation. See Figs. 5–6.

Remark 12. When the regularization parameter gets larger than 1/8, the equation loses its forward–backward and degenerate character and possesses global classical solutions.

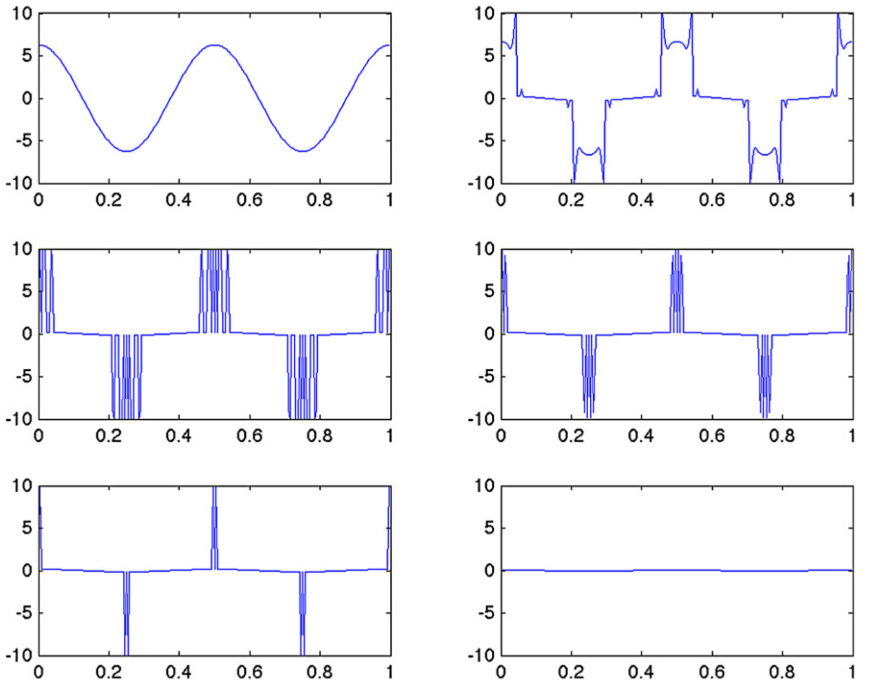

Fig. 6. Long time behavior of the gradient of a solution to (1.3).

Remark 13. For transcritical initial data there are effectively two times scales of evolution. In the forward and fast regime, the solution quickly tends to become constant, whereas, in the backward and slow regime, oscillations appear which survive for a long time because they cannot be readily damped by the equation as the solution carries very little left-over energy. The oscillations are caused by the solution attempting to flee the unstable regime by developing gradients of preferred small and large size. As δ tends to zero, the preferred slope sizes approach the values 0 and ∞, respectively. This is precisely the phenomenon of staircasing observed in numerical simulations of PME.

Remark 14. These theoretical results show that a typical solution to (1.3) never develops jumps, thus does not suffer from staircasing. Numerical experiments clearly indicate that the latter is replaced, as soon as $\delta > 0$ by a milder form of infinitesimal behavior, which could appropriately be called micro-structured ramping. Solutions with transcritical initial data will develop regions of instability in between fast forming macro-plateaus where oscillations of higher and higher frequency (in the sense of approximating sequences) are generated in an attempt to escape the unstable regime. It is also apparent that the oscillations are between two preferred states with almost vanishing and ever growing slope (in terms of δ), respectively. This behavior is illustrated in Fig. 7 where the horizontal lines represent approximations to the locations where the energy function switches convexity and are simultaneously upper or lower bounds for the preferred gradient values. This behavior is captured theoretically by the concept of Young measure solution and by the location of the support of the gradient Young measure associated to the solution.

Acknowledgments

The author would like to thank Maria Colombo, Marina Ghisi, and Massimo Gobbino for interesting discussions and for drawing his attention to the validity of inequality (2.6).

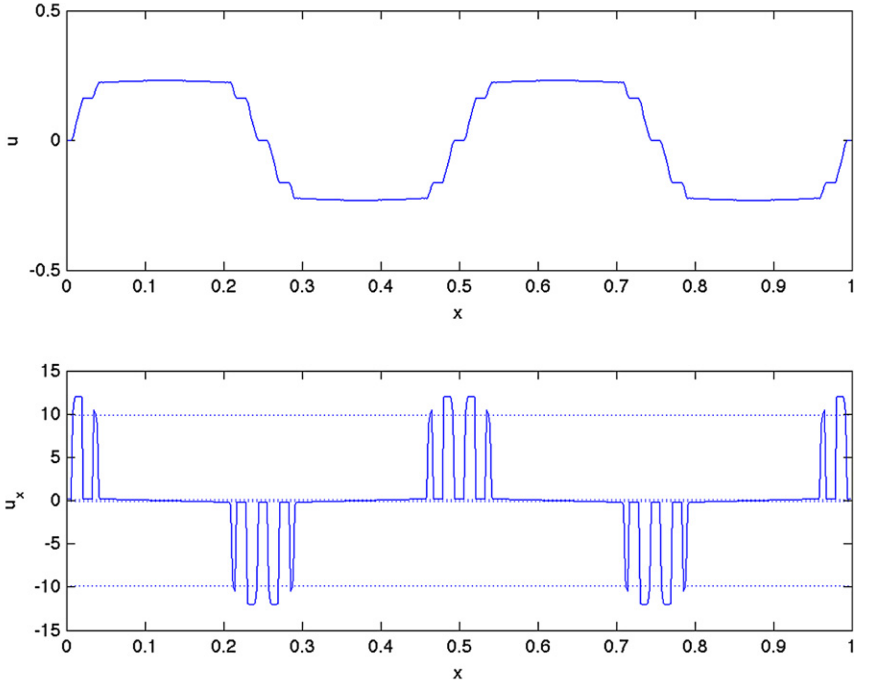

Fig. 7. The ramping phenomenon which replaces staircasing of PME and to which it converges in the vanishing limit of the regularization.

This research was partially supported by the National Science Foundation under award number DMS-0712875.

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