A New Well-posed Nonlinear Nonlocal Diffusion

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Abstract

A new nonlinear diffusion is proposed and analyzed. It is characterized by a nonlocal dependence in the diffusivity which manifests itself through the presence of a fractional power of the Laplacian. The equation is related to the well-known and ill-posed Perona-Malik equation of image processing. It shares with the latter some of its most cherished features while being well-posed. Local and global well-posedness results are presented along with numerical experiments which illustrate its interesting dynamical behavior mainly due to the presence of a class of meta-stable non-trivial equilibria.

Key words: Nonlinear nonlocal diffusion, well-posedness, noise reduction, Perona-Malik.

1 Introduction

In this paper a new nonlinear nonlocal diffusion model is proposed and analyzed which can be used for noise reduction in image processing [1]. One of its main features is that it admits a non-trivial class of equilibria. They are at the origin of the interesting dynamical behavior of solutions which makes the

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model particularly appealing as a denoising tool. The model reads

$$\begin{cases} u_t = \frac{1}{1+[(-\triangle)^{1-\varepsilon}u]^2}\triangle u & \text{in } \Omega \text{ for } t>0\,, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t>0\,, \\ u(0) = u_0 & \text{in } \Omega \text{ for } t=0\,. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is an open domain with smooth boundary $\partial\Omega$ and ν is the unit outer normal vector. In applications, the domain is typically a square or rectangle but, in spite of the assumption on Ω made here, the results obtained in this paper will be clearly applicable to that case, too. There is a vast literature concerning nonlinear anisotropic diffusions with application to image processing which date back to the seminal paper by Perona and Malik, who, in [2], consider a discrete version of the equation

$$\begin{cases} u_t = \operatorname{div}\Big(\frac{1}{1+|\nabla u|^2}\nabla u\Big) & \text{in } \Omega \text{ for } t>0\,, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t>0\,, \\ u(0) = u_0 & \text{in } \Omega \text{ for } t=0\,. \end{cases} \tag{1.2}$$

While discretizations of (1.2) yield a quite effective denoising tool, the equation itself is mathematically ill-posed as has been pointed out in [3] and its discretization exhibits some practical shortcomings. By computing the divergence in (1.2) one obtains

$$\operatorname{div}\Big(\frac{1}{1+|\nabla u|^2}\nabla u\Big) = \frac{1}{1+|\nabla u|^2}\Big[\triangle u - 2\frac{\nabla u^T D^2 u \nabla u}{1+|\nabla u|^2}\Big]\,.$$

It follows that, if u has a sharp gradient across a hyper-surface, diffusion in normal direction can reverse its sign. Thus (1.2) is a so-called forward-backward diffusion equation and this is precisely the structure which has been exploited in [3] to prove ill-posedness in the one dimensional case.
Equation (1.2) can also be viewed as the gradient flow associated to the energy functional

$$E(u) = \int_\Omega \log\Big(1+|\nabla u|^2\Big)\,dx \tag{1.3}$$

which is clearly not convex (except in a neighborhood of $\nabla u \equiv 0$). Taking this point of view and using variational techniques, the authors of [4,5] prove instability and the existence of infinitely many weak solutions. In the region of convexity of (1.3) it can be shown [3] that any initial condition possessing small gradients yields a global solution possessing small gradients which eventually converges to a constant steady state. Clearly this is not of much interest in image processing where typical images have sharp gradients, called edges or step edges.
Solutions to discretizations of (1.2), which usually follow the structure

$$\begin{cases} \frac{u^{n+1}-u^n}{h} = \operatorname{div}\Big(\frac{1}{1+|\nabla u^n|^2}\nabla u^{n+1}\Big)\,, \\ u^0 = u_0\,, \end{cases} \tag{1.4}$$

can be stably computed and exhibit preservation of sharp edges over sizeable time intervals. This is in stark contrast with the theoretical ill-posedness result and has been called *the Perona-Malik paradox* in [3]. It should be pointed out that equation (1.2) also has some practical drawbacks. Among them are the preservation of artificial edges caused by noise and the phenomenon known as "stair-casing" because of its visual appearance (see Fig. 1 and also [3]).
In order to obtain well-posed models a series of regularizations have been proposed and analyzed in the literature. They fall into two main categories: Spatial and spatio-temporal regularizations. Spatial regularizations are obtained by smoothing the argument of the nonlinearity by a C^∞-kernel G_σ, thus leading to the equation

$$\begin{cases} u_t = \operatorname{div}\Big(\frac{1}{1+|\nabla G_\sigma * u|^2}\nabla u\Big)\,, \\ \partial_\nu u = 0\,, \\ u(0) = u_0\,. \end{cases}$$

Typically G_σ is a Gaussian and σ determines the scale beyond which regularization occurs. This approach goes back to [6] but also see [7,8]. More recently, a number of researchers have begun considering spatio-temporal regularizations and/or time relaxations of (1.2). In this case the argument of the nonlinearity is regularized by a space-time convolution or substituted with relaxed versions of it as in [9–11]. Relaxation models with no spatial regularizations have been considered in [12,13]. The latter are limiting cases of the model considered in [9]. In [14], the author tries to mimic (1.4) and proposes a purely temporal regularization with a non-smooth kernel. This leads to a delayed Perona-Malik equation. The new equation is locally well-posed and seems to introduce significantly less blurring than the more conventional regularization techniques. Finally a somewhat different but still PDE-based approach was proposed by Rudin and Osher [15]. It uses the equation

$$u_t = |u_x| F(u_{xx})\,,\ u(0) = u_0\ (sF(s) \geq 0)$$

which allows for the formation of sharp edges.

One of the possible reasons why the Perona-Malik equation works well in practice might be that characteristic functions are, at least formally in some sense, stationary solutions of the equation, a fact that has been observed in [3]. This is a desirable property for any denoising equation to have but is not satisfied by any of the known regularizations of the Perona-Malik equation and motivated [16] to propose a regularization for which it could be proved that this is the case at least in an approximate sense quantified in terms of the regularization parameter.

The equation proposed in this paper is new and possesses a number of welcome and desirable properties from both the theoretical and the practical point of view. First of all it is well-posed and admits piecewise affine functions as stationary solutions (in one space dimension). These features make working with discretizations of it very safe and effective. Secondly, it allows for a natural, fft-based, efficient and stable pseudo-spectral discretization (cf. Section 3 of this paper and [1]). Experiments performed with it demonstrate its effectiveness at noise-reduction with remarkable preservation of sharp edges. It also appears to avoid known shortcomings of discretizations of the Perona-Malik equation itself. For exhaustive two dimensional numerical demonstrations with (1.1) and a related nonlocal nonlinear diffusion in divergence form it is referred to [1].

It should be observed that methods other than PDE-based nonlinear diffusions have been studied and implemented in the context of noise reduction. The interested reader is referred to the survey paper [17] and the references cited therein for a wider perspective on the subject.

The paper is organized as follows. The next section is devoted to proving that (1.1) is well-posed in the context of strong solutions for any $\varepsilon > 0$. In Section 2.2 it is also shown to be globally well-posed in one space dimension on the more stringent assumption that $\varepsilon > 3/4$. In this case solutions are proved to converge to a trivial steady-state. In Section 2.3 the relation between equation (1.1) and the Perona-Malik equation (1.2) is investigated in the one dimensional setting where it can be shown to be a new type of spatial regularization of an equivalent formulation of (1.2). It should be observed, however, that this regularization is essentially milder than any of the ones previously considered in the literature as it does not discriminate between different scales. It is best thought of as a reduction of the nonlinearity intensity. In Section 2.4 it is proved that continuous piecewise affine functions are legitimate steady states of the one-dimensional version of (1.1) which correspond to piecewise constant "solutions" of the original Perona-Malik equation. The one dimensional numerical experiments performed in Section 3 underscore the theoretical predictions about the interesting dynamical behavior engendered by the equation and caused by the fact that piecewise affine functions are indeed stationary solutions. The latter, however, seem to determine the behavior of solutions only in the short and medium term, whereas diffusion seems to take over in the long run, eventually driving solutions to a trivial steady-state. Piecewise affine functions could therefore be labeled as "meta-stable" solutions. Two two-dimensional numerical experiments are also shown for "real images" at the very end of the paper; they corroborate the claims made concerning the practicality and efficacy of the method.

2 Well-posedness

2.1 Local Existence

Local well-posedness is obtained exploiting the quasi-linear structure of the equation which follows from $\varepsilon \in (0,1]$. This can be done in the framework of maximal regularity for parabolic problems. Some notation is needed to simplify the presentation.
Let E_0 and E_1 be Banach spaces and assume that

$$E_1 \overset{d}{\hookrightarrow} E_0$$

where the letter "d" indicates density of the embedding. For a linear operator

$$A : \mathrm{dom}(A) \subset E_0 \longrightarrow E_0$$

with $D(A) = E_1$, consider the abstract Cauchy problem

$$\begin{cases} \dot{u} + Au = f\ ,\ t > 0\ , \\ u(0) = u_0\ , \end{cases} \tag{2.1}$$

in the Banach space E_0. Given $p \in (1, \infty)$, A is said to enjoy maximal L_p-regularity on $J_T = [0, T]$ if (2.1) possesses a unique solution $u : J_T \to E_0$ satisfying

$$u \in \mathrm{L}_p(J_T, E_1) \cap \mathrm{W}^1_p(J_T, E_0) =: \mathbb{E}_{1,p}(J_T) =: \mathbb{E}_{1,p}$$

for each $(u_0, f) \in (E_0, E_1)_{1-1/p,p} \times \mathrm{L}_p(J_T, E_0) =: \gamma\mathbb{E}_{1,p} \times \mathbb{E}_{0,p}$. Hereby $(E_0, E_1)_{1-1/p,p}$ denotes the standard real interpolation functor (see [18]). $\gamma\mathbb{E}_{1,p}$ is the natural trace space associated to $\mathbb{E}_{1,p}$ and the Cauchy problem (2.1) and

$$\mathbb{E}_{1,p}(J_T) \hookrightarrow \mathrm{C}\Big(J_T, (E_0, E_1)_{1-1/p,p}\Big)\ , \tag{2.2}$$

as follows from [19, Theorem III.4.10.2]. The property of maximal regularity does not depend on the the interval length $T < \infty$. Maximal regularity implies that

$$\|\dot{u}\|_{\mathrm{L}_p(J_T,E_0)} + \|u\|_{\mathrm{L}_p(J_T,E_1)} \le c\Big(\|u_0\|_{(E_0,E_1)_{1-1/p,p}} + \|f\|_{\mathrm{L}_p(J_T,E_0)}\Big)$$

or, more suggestively that

$$\Big(\partial_t + A, \gamma_0\Big) \in \mathcal{L}_{is}\Big(\mathbb{E}_{1,p} \times \gamma\mathbb{E}_{1,p}, \mathbb{E}_{0,p}\Big)$$

as pointed out in [20, Proposition 3.1]. The collection of operators $A \in \mathcal{L}(E_1, E_0)$ enjoying maximal regularity is denoted by $\mathcal{MR}_p(E_0, E_1)$ and is endowed with the topology induced by $\mathcal{L}(E_1, E_0)$.

If $A \in \mathrm{L}_\infty\Big(J_T, \mathcal{MR}_p(E_0,E_1)\Big)$, maximal regularity for the corresponding non autonomous Cauchy problem is defined just as in the autonomous case considered above and the collection of operators $A \in \mathrm{L}_\infty\Big(J_T, \mathcal{MR}_p(E_0,E_1)\Big)$ enjoying maximal regularity is denoted by $\mathcal{MR}_p(J_T,E_0,E_1)$.
Recently maximal L_p-regularity has been obtained on the mild assumption that

$$A \in \mathrm{C}\Big(J_T, \mathcal{MR}_p(E_0,E_1)\Big)\,.$$

For two different accounts of this it is referred to [21] and [20]. As for the quasi-linear Cauchy problem

$$\begin{cases} \dot u + A(u)u = F(u)\,, t\in J_T\,,\\ u(0)=u_0\,, \end{cases} \tag{2.3}$$

a variety of results are available in the literature [22–26]. Here the recent and particularly elegant result contained in [27] is used. It gives existence of a unique maximal solution $u: J_{max} \to E_0$ to (2.3) with $J_{max}=[0,T_{max}]\subset J_T$ and

$$u\in \mathrm{L}_p(J_{max},E_1)\cap \mathrm{W}^1_p(\dot J_{max},E_0)\,,$$

where $\dot J_{max} = [0,T_{max})$. A solution is said to be maximal if it cannot be extended to a larger time interval. It is obtained by iterating a fixed-point existence argument with ever diminishing interval length. Furthermore global existence is obtained as soon as it can be shown that

$$u \in \mathbb{E}_{1,p}(J_{max}) \text{ or } A(u)\in \mathcal{MR}_p(J_{max},E_0,E_1)\,. \tag{2.4}$$

The result rests on the assumption that the nonlinearities satisfy

$$A \in \mathrm{C}^{-1}_{Volt}\Big(\mathbb{E}_{1,p}(J_T), \mathcal{MR}_p(J_T,E_0,E_1)\Big)\,, \tag{2.5}$$

$$F \in \mathrm{C}^{-1}_{Volt}\Big(\mathbb{E}_{1,p}(J_T); \mathrm{L}_q(J_T,E_0),\mathbb{E}_{0,p}\Big)\,. \tag{2.6}$$

It is not necessary to dwell on the precise meaning of the second assumption here since in (1.1) the right-hand-side vanishes. The condition on A means that the nonlinearity should have the so-called Volterra property and otherwise be bounded and uniformly Lipschitz continuous on bounded sets. The Volterra property is only relevant for nonlinearities allowed to be non local in time but is irrelevant for local ones such as in (1.1), where the non locality is in space. For reasons which will become clear later it is convenient to consider also the case where the Neumann condition in (1.1) is replaced by a homogeneous Dirichlet condition. In order to verify that (1.1) indeed satisfies the conditions of the above theorem some more objects need to be identified. Let $E_0=\mathrm{L}_p(\Omega)$. Then

$$E_{1,L} = \mathrm{dom}(-\triangle_L) = \begin{cases} \mathrm{W}^2_{p,L}(\Omega) = \Big\{u\in \mathrm{W}^2_p(\Omega)\,\Big|\, \partial_\nu u = 0\Big\} & \text{if } L=N\,,\\ \mathrm{W}^2_{p,L}(\Omega) = \Big\{u\in \mathrm{W}^2_p(\Omega)\,\Big|\, \gamma_{\partial\Omega} u = 0\Big\} & \text{if } L=D\,, \end{cases} \tag{2.7}$$

where the subscripts N and D indicate that the L_p-realization of the Laplacian with homogeneous Neumann and Dirichlet condition are considered, respectively. In order to simplify notation the nonlinear diffusivity $\frac{1}{1+|A_L^{1-\varepsilon}u|^2}$ appearing in (1.1) is denoted by $a_\varepsilon(u)$. Hereby A_L denotes $-\triangle_L$ for $L = N,\ D$. In the sequel the subscript L denotes either N or D without further mention.

Lemma 1 *Let $\varepsilon \in (0,1]$ and assume that $p > \frac{2+n}{2\varepsilon}$ depending on the dimension $n = 1, 2$. Given $u \in \mathbb{E}_{1,p,L}$ and $T > 0$, one has that*

$$-a_\varepsilon(u)\triangle_L \in \mathrm{C}\Big(J_T, \mathcal{MR}_p(E_0, E_{1,L})\Big) \subset \mathcal{MR}_p\big(J_T, E_0, E_{1,L}\big) .$$

PROOF. It is known (see [28]) that

$$(E_0, E_{1,N})_{1-1/p,p} \doteq \mathrm{W}_{p,N}^{2-2/p}(\Omega) = \begin{cases} \Big\{u \in \mathrm{W}_p^{2-2/p}(\Omega) \,\Big|\, \partial_\nu u = 0\Big\}, & 2 - \frac{2}{p} > 1 + \frac{1}{p}, \\ \mathrm{W}_p^{2-2/p}(\Omega), & 2 - \frac{2}{p} < 1 + \frac{1}{p}, \end{cases}$$

and

$$(E_0, E_{1,D})_{1-1/p,p} \doteq \mathrm{W}_{p,D}^{2-2/p}(\Omega) = \begin{cases} \Big\{u \in \mathrm{W}_p^{2-2/p}(\Omega) \,\Big|\, \gamma_{\partial\Omega} u = 0\Big\}, & 2 - \frac{2}{p} > \frac{1}{p}, \\ \mathrm{W}_p^{2-2/p}(\Omega), & 2 - \frac{2}{p} < \frac{1}{p}. \end{cases}$$

Therefore (2.2) gives

$$\mathbb{E}_{1,p,L} \hookrightarrow \mathrm{C}\Big(J_T, \mathrm{W}_{p,L}^{2-2/p}(\Omega)\Big) . \tag{2.8}$$

Furthermore

$$A_L^{1-\varepsilon} \in \mathcal{L}\Big(\mathrm{H}_{p,L}^{2-2/p}(\Omega), \mathrm{H}_{p,L}^{2\varepsilon-2/p}(\Omega)\Big) ,$$

where the function spaces are Bessel potential spaces (see [18]). This follows from the fact that the domains of the fractional powers of the Laplacian are characterized by the complex interpolation functor (see [19,29])

$$\mathrm{dom}(A_L^\rho) \doteq [E_0, E_{1,L}]_\rho = \mathrm{H}_{p,L}^{2\rho}(\Omega)$$

for $\rho \in (0,1)$ provided the operator has bounded imaginary powers. This was proved by Seeley [29] for elliptic operators on domains with homogeneous boundary conditions and smooth coefficients. In this case it is also known that the reiteration theorem of [30] applies to give

$$[E_{\alpha,L}, E_{\beta,L}]_\theta = E_{(1-\theta)\alpha+\theta\beta,L}\ ,\ \theta \in (0,1) ,$$

where $E_{\gamma,L} = [E_0, E_{1,L}]_\gamma$ for $\gamma = \alpha, \beta$ and $0 < \alpha \le \beta < 1$. It follows that

$$A_L^\rho \in \mathcal{L}\Big(\mathrm{H}_{p,L}^{s+2\rho}(\Omega), \mathrm{H}_{p,L}^{s}(\Omega)\Big)$$

for $s \geq 0$ such that $0 < s + 2\rho \leq 2$. Next the almost reiteration theorem [31] gives

$$\mathrm{W}^t_{p,L}(\Omega) \hookrightarrow \mathrm{H}^{t-\delta}_{p,L}(\Omega)$$

whenever $0 < \delta$. In summary one has that

$$A_L^{1-\varepsilon} \in \mathcal{L}\Big(\mathrm{W}^{2-2/p}_{p,L}(\Omega), \mathrm{H}^{2\varepsilon-2/p-\delta}_{p,L}(\Omega)\Big)$$

provided $\delta << 1$. The assumption combined with a classical embedding theorem therefore implies that

$$\mathrm{H}^{2\varepsilon-2/p-\delta}_{p,L}(\Omega) \hookrightarrow \mathrm{C}^{2\varepsilon-\delta-\frac{2+n}{p}}(\overline{\Omega})$$

since $\delta << 1$ can be chosen such that $2\varepsilon - \delta - \frac{2+n}{p} > 0$. Next, results on the behavior of Nemitskii operators on spaces of Hölder continuous functions [32] entail that the map

$$v \mapsto \frac{1}{1+v^2}\,,\ \mathrm{C}^\alpha(\overline{\Omega}) \to \mathrm{C}^\alpha(\overline{\Omega})$$

is well-defined, bounded and uniformly Lipschitz continuous on bounded subsets. In order to apply the results of [32] it is enough to check that

$$[x \mapsto \frac{1}{1+x^2}] \in \mathrm{BUC}^2(\mathbb{R}, \mathbb{R})\,,$$

which is clearly satisfied. Putting everything together one gets that

$$\mathbb{E}_{1,p,L} \ni u \mapsto \frac{1}{1+|A_L^{1-\varepsilon}u|^2} =: a_\varepsilon(u) \in \mathrm{C}\Big(J_T, \mathrm{C}^\rho(\overline{\Omega})\Big)$$

for a positive ρ and, consequently, that

$$[t \mapsto a_\varepsilon\big(u(t)\big)\triangle_L] \in \mathrm{C}\Big(J_T, \mathcal{L}(E_{1,L}, E_0)\Big)\,. \tag{2.9}$$

The Hölder continuity of $a_\varepsilon\big(u(t)\big)$ is enough to apply a result by Hieber and Prüß [33] which combines generation of analytic semigroups [25] with Gaussian bounds [34] to obtain maximal regularity (see [21, Section 4]). Their result gives

$$a_\varepsilon\big(u(t)\big)\triangle_L \in \mathcal{MR}_p(E_0, E_{1,L})$$

for any fixed $t \in J_T$. This, together with (2.9), implies via [20, Theorem 7.1] that

$$a_\varepsilon(u)\triangle_L \in \mathcal{MR}_p(J_T, E_0, E_{1,L})\,.$$

Finally, the Lipschitz continuity follows from [32] since, for $R > 0$, one has

$$\begin{aligned}
\|a_\varepsilon\big(u(t)\big) - a_\varepsilon\big(v(t)\big)\|_{\mathrm{C}^\rho(\overline{\Omega})} &\le L\|A^{1-\varepsilon}\big[u(t) - v(t)\big]\|_{\mathrm{C}^\rho(\overline{\Omega})} \\
&\le L\|A^{1-\varepsilon}\big[u(t) - v(t)\big]\|_{\mathrm{W}_p^{2\varepsilon-\delta-\frac{2+n}{p}}(\Omega)} \le L\|u(t) - v(t)\|_{\mathrm{W}_p^{2-2/p}(\Omega)} \\
&\le L\|u - v\|_{\mathrm{C}\left(J_T, \mathrm{W}_p^{2-2/p}(\Omega)\right)} \le L\|u - v\|_{\mathbb{E}_{1,p}}\, ,\ u, v \in \mathbb{B}_{\mathbb{E}_{1,p,L}}(0, R)\, .
\end{aligned}$$

Thus

$$\|\big[a_\varepsilon(u) - a_\varepsilon(v)\big]\triangle_L\|_{\mathrm{C}\left(J_T, \mathcal{L}(E_{1,L}, E_0)\right)} \le L\|u - v\|_{\mathbb{E}_{1,p,L}}$$

on bounded subsets of $\mathbb{E}_{1,p,L}$ and the condition of [27, Theorem 2.1] is indeed satisfied.

Thus the following theorem is obtained.

Theorem 2 *Let $\varepsilon \in (0, 1]$ and assume that $u_0 \in \mathrm{W}_{p,N}^{2-2/p}(\Omega)$ and that $p > \frac{2+n}{2\varepsilon}$. Then equation* (1.1) *has a unique maximal solution $u : J_{max} \to \mathrm{L}_p(\Omega)$ with*

$$u \in \mathrm{L}_p\Big(J_{max}, \mathrm{W}_{p,N}^2(\Omega)\Big) \cap \mathrm{W}_p^1\Big(\dot{J}_{max}, \mathrm{L}_p(\Omega)\Big)\, .$$

Furthermore if

$$a_\varepsilon(u)\triangle \in \mathcal{MR}_p\Big((J_{max}, \mathrm{L}_p(\Omega), \mathrm{W}_{p,N}^2(\Omega)\Big)$$
$$\text{or } u \in \mathrm{L}_p\Big(J_{max}, \mathrm{W}_{p,N}^2(\Omega)\Big) \cap \mathrm{W}_p^1\Big(J_{max}, \mathrm{L}_p(\Omega)\Big)\, ,$$

then the solution is global in time. If homogeneous Dirichlet conditions are imposed, the result remains true with N replaced by D.

Next it is shown that solutions emanating from smooth initial data are actually smooth.

Proposition 3 *Assume that $u_0 \in \mathrm{C}^\infty(\overline{\Omega})$ and that it satisfies compatibility conditions to all orders. Then the solution u satisfies*

$$u \in \mathrm{C}^\infty\Big(\dot{J}_{max} \times \overline{\Omega}\Big)$$

PROOF. The embedding (2.2) implies that

$$u \in \mathrm{C}^\rho\Big(\dot{J}_{max}, \mathrm{W}_{p,L}^{2-2\rho}(\Omega)\Big) \text{ for any } p > 1 \text{ and some } \rho > 0\, ,$$

provided $2 - 2\rho < 2 - 2/p$. To see this, first notice that

$$\|u(t) - u(s)\|_{\mathrm{W}_p^{2-2\rho}(\Omega)} \le c\|u(t) - u(s)\|_{\mathrm{L}_p(\Omega)}^{1-\theta}\|u(t) - u(s)\|_{\mathrm{W}_p^{2-2/p}(\Omega)}^{\theta}$$

by the almost reiteration theorem. Then Hölder continuity in time follows from

$$\|u(t) - u(s)\|_{\mathrm{L}_p(\Omega)} \leq \int_s^t \|\dot{u}(\tau)\|_{\mathrm{L}_p(\Omega)}\, d\tau$$

and Hölder's inequality. Now, since $p > 1$ can be chosen arbitrarily large, it follows that

$$\mathrm{W}^{2(\varepsilon-\rho)}_{p,L}(\Omega) \hookrightarrow \mathrm{C}^{\delta}(\overline{\Omega})$$

since ρ, p can be chosen as to ensure that $\delta = 2(\varepsilon - \rho) - \frac{n}{p} > 0$. Then by [32]

$$\frac{1}{1 + |A_L^{1-\varepsilon} u|^2} \in \mathrm{C}^{\delta, \frac{\delta}{2}}(\dot{J}_{max} \times \overline{\Omega})$$

and classical regularity results [35] entail that

$$u \in \mathrm{C}^{2+\delta, 1+\frac{\delta}{2}}(\dot{J}_{max} \times \overline{\Omega})\,.$$

Since $A_L^{1-\varepsilon}$ is lower order than A_L, this boot-strapping argument can be iterated in order to improve the spatial and time regularity of the solution again by means of classical regularity theory as long as compatibility conditions are satisfied [35].

2.2 Long Time Behavior

It is an easy consequence of the maximum principle that a solution of (1.1) satisfies

$$\inf_{x \in \overline{\Omega}} u_0(x) \leq u(t,x) \leq \sup_{x \in \overline{\Omega}} u_0(x) \tag{2.10}$$

as long as the solution exists. This is a welcome property from the point of view of image processing where a gray-scale image is a function assuming integer values in the interval $[0, 255]$. The equation also possesses the Lyapunov functional

$$L(u) := \int_\Omega |\nabla u|^2\, dx \tag{2.11}$$

as follows from

$$\frac{d}{dt} L(u) = -2 \int_\Omega \frac{|\triangle u|^2}{1 + |A_N^{1-\varepsilon} u|^2}\, dx\,, \tag{2.12}$$

which is easily obtained multiplying the equation by $\triangle u$ and integrating over the domain. Thus the following estimate is also obtained

$$\int_\Omega |\nabla u|^2\, dx + \int_0^t \int_\Omega \frac{|\triangle u|^2}{1 + |A_N^{1-\varepsilon} u|^2}\, dx d\tau = \int_\Omega |\nabla u_0|^2\, dx\,. \tag{2.13}$$

If $a_\varepsilon(u) = (1 + |A_N^{1-\varepsilon} u|^2)^{-1}$ could be estimated from below, (2.13) together with (2.10) would imply that

$$u \in \mathrm{L}_2\Big(J_{max}, \mathrm{W}^2_{2,N}(\Omega)\Big) \tag{2.14}$$

and, using the equation, that

$$u \in \mathrm{W}_2^1\Big(J_{max}, \mathrm{L}_2(\Omega)\Big)\,. \tag{2.15}$$

Thus, if $n = 1$ and $p = 2$, the solution exists globally for ε large enough provided a point-wise upper bound for $|A_N^{1-\varepsilon}u|$ can be obtained.

Theorem 4 *If $n = 1$ and $\varepsilon > 3/4$, the solution to* (1.1) *exists globally and converges to a trivial steady-state.*

PROOF. It is easy to check that $p = 2$ is admissible in Theorem 2 on the given assumptions. Owing to the regularity of the solution it is seen that $v = u_x$ satisfies the equation

$$\begin{cases} v_t - \Big(a_\varepsilon(u)v_x\Big)_x = 0\,,\ t \in [0, T)\,, \\ v(t, 0) = v(1, 0) = 0\,,\ t \in [0, T)\,, \\ v(0, \cdot) = u_0'\,. \end{cases}$$

for any $T < T_{max}$. The classical maximum principle then implies

$$\inf_{x\in\overline{\Omega}} u_0'(x) \le v(t, x) = u_x(t, x) \le \sup_{x\in\overline{\Omega}} u_0'(x) \tag{2.16}$$

which gives a point-wise upper bound for u_x. It follows that $|A_N^{1-\varepsilon}u|$ can also be estimated point-wise and the global existence follows from Theorem 2 and the considerations preceding this theorem. Next it follows from (2.16) and (2.12) that

$$\frac{d}{dt}\int_\Omega |u_x|^2\,dx \le -c\int_\Omega |u_{xx}|^2\,dx \le -c\int_\Omega |u_x|^2\,dx\,,$$

where the second inequality is a consequence of Poincaré's inequality in view of the boundary conditions. Thus

$$\int_\Omega |u_x|^2\,dx \le Ce^{-ct}\,,\ t \ge 0\,.$$

Convergence to a trivial steady-state follows.

It is also observed that local extrema of the first derivative of a solution are not enhanced. In fact if $x_0 \in (0, 1)$ is a point of maximum [minimum] of $u_x(t, \cdot)$, it follows that

$$u_{tx}(t, x_0) = a_\varepsilon(u)_x(t, x_0)u_{xx}(t, x_0) + a_\varepsilon(u)(t, x_0)u_{xxx}(t, x_0) \le 0\,[\ge 0]\,,$$

since clearly $u_{xx}(t, x_0) = 0$ and $u_{xxx}(t, x_0) \le 0$ $[\ge 0]$.
Numerical experiments suggest that second derivatives, however, can increase

significantly for smooth initial data. Since equation (1.1) is very close to a backward-forward equation (obtained for $\varepsilon = 0$), this should not come as a surprise, since, for ε very small, arbitrarily, but at most finitely, many modes can go linearly unstable.

2.3 Relation to the Perona-Malik Equation

In its one dimensional formulation, the Perona-Malik equation reads

$$\begin{cases} u_t - \left(\frac{1}{1+u_x^2} u_x\right)_x = 0 & \text{in } (0,1) \text{ for } t > 0\,, \\ u_x(t,0) = 0 = u_x(t,1) & \text{for } t > 0\,, \\ u = u_0 & \text{at } t = 0 \text{ in } [0,1]. \end{cases} \tag{2.17}$$

Introducing $v(t,x) = \int_0^x u(t,y)\,dy$ as a new unknown, it is easily seen that it satisfies

$$\begin{cases} v_t - \frac{1}{1+v_{xx}^2} v_{xx} = 0 & \text{in } (0,1) \text{ for } t > 0\,, \\ v(t,0) = 0\,,\ v(t,1) = \int_0^1 u_0(y)\,dy & \text{for } t > 0\,, \\ v(0,\cdot) = \int_0^{\cdot} u_0(y)\,dy & \text{in } [0,1]\,, \end{cases} \tag{2.18}$$

since it is readily verified that the average of u_0 is preserved over time using (2.17). By further defining $w(x) = v(x) - x\int_0^1 u_0(y)\,dy\,,\ x \in [0,1]\,,$ it follows that

$$\begin{cases} w_t - \frac{1}{1+w_{xx}^2} w_{xx} = 0 & \text{in } (0,1) \text{ for } t > 0\,, \\ w(t,0) = 0 = w(t,1) & \text{for } t > 0\,, \\ w(0,\cdot) = \int_0^{\cdot} u_0(y)\,dy - (\cdot)\int_0^1 u_0(y)\,dy & \text{in } [0,1]\,. \end{cases} \tag{2.19}$$

Conversely, any solution of (2.18) leads to a solution of (2.17) by setting

$$u(t,x) = v_x(t,x)\,,\ x \in [0,1]\,,\ t \geq 0\,.$$

The evolution equation is clearly satisfied whereas for the boundary conditions the following argument is needed. The function

$$\tilde{v}(t,x) = \int_0^x u(t,y)\,dy\,,\ x \in [0,1]\,,$$

satisfies

$$\tilde{v}_t - \frac{1}{1+\tilde{v}_{xx}^2}\tilde{v}_{xx} = -\frac{u_x(t,0)}{1+u_x^2(t,0)}\,.$$

Since $\tilde{v} = v$ by $v(t,0) = 0$ and since v satisfies (2.18), it follows that $\frac{u_x(t,0)}{1+u_x^2(t,0)} = 0$ and therefore $u_x(t,0) = 0$. Now

$$\int_0^1 u(t,y)\,dy = v(t,1) - v(t,0) = \int_0^1 u_0(y)\,dy$$

implies that

$$0 = \int_0^1 u_t(t,y)\,dy = \frac{u_x(t,1)}{1+u_x^2(t,1)} - \frac{u_x(t,0)}{1+u_x^2(t,0)} = \frac{u_x(t,1)}{1+u_x^2(t,1)}$$

and therefore that $u_x(t,1) = 0$, too. Notice that if u is a piecewise constant function, then v is a continuous piecewise affine function. Thus piecewise affine functions play the same role for (2.19) as piecewise constant functions do for (2.17). The latter were observed to be some kind of stationary solutions for the one dimensional Perona-Malik equation in [3].
Equation (2.19) is fully nonlinear and clearly presents the same analytical difficulties as the original Perona-Malik equation. It is, however, closely related to (1.1). The latter can in fact be obtained from it by slightly reducing the nonlinearity intensity (which also amounts to a very mild regularization) from the original to

$$a_\varepsilon(u) := \Big(1 + [A_D^{1-\varepsilon} u]^2\Big)^{-1}$$

where it is recalled that $A_D = -\triangle_D$ denotes the Dirichlet Laplacian because of the boundary conditions considered. This is perfectly analogous to (1.1). Notice that the intensity of the nonlinearity is barely reduced. It is therefore to be expected that derivatives of solutions to the modified equation

$$\begin{cases} u_t - a_\varepsilon(u) A_D u = 0\,, \\ u(0) = u_0\,, \end{cases} \tag{2.20}$$

with $\varepsilon > 0$ small behave similarly to the solutions of (2.17), at least at the discrete level, when $\varepsilon = 0$ in which case the original (1.2) is recovered. At the continuous level (1.2) is ill-posed and such a claim is not very meaningful. The use of equation (2.20) (or (1.1)) is thus likely to provide a viable practical denoising tool while still preventing blurring as the regularization is significantly milder than more traditional ones.
Observe that (2.20) is locally well-posed for $\varepsilon \in (0,1]$ and globally well-posed for $\varepsilon > 3/4$. The proofs given in Section 2.1 in fact include the case of homogeneous Dirichlet boundary conditions. Notice that the solution of (2.20) also satisfies

$$\frac{d}{dt}\int_0^1 v\,dx = 0 \text{ and } \frac{1}{2}\frac{d}{dt}\int_0^1 |v|^2\,dx = -\int_0^1 \frac{|v_x|^2}{1+[A_D^{1-\varepsilon}u]^2}\,dx \tag{2.21}$$

for $v = u_x$ and a smooth solution u of (2.20). The choice of boundary conditions in (1.1) is motivated by its effectiveness in practical implementations, where it is empirically observed to produce the best results. It should also be pointed out that ε would typically be chosen slightly larger than $1/2$ in two dimensional numerical experiments. Thus the considerations of this section are really limited to the one dimensional context and only provide some heuristic motivation for suggesting (1.1) as a denoising tool.

The trivial function $u \equiv 0$ is clearly a stationary solution of (2.20). Similarly constant functions $u \equiv c$ are equilibria for (1.1) for $n = 1$, that is, for the corresponding problem with homogeneous Neumann boundary conditions.
An interesting aspect of equations (2.20) and (1.1) is, however, that they admit a family of non-trivial steady-states. These are continuous piecewise affine functions and their presence engenders a non-trivial dynamical behavior. This will be clearly illustrated by the numerical experiments performed in the last section of this paper which seem to indicate that they arguably play the role of *meta-stable* solutions. Generic solutions tend in fact to flow in their vicinity while eventually converging to a trivial state.
The long time behavior of global smooth solutions to (2.20) or to (1.1) is controlled by the Lyapunov functionals (2.11) and (2.21), respectively. They imply convergence to trivial steady-states in that case. This happens for $\varepsilon > 3/4$ as observed in Section 2.2. In general, however, blow-up in finite time can not be ruled out.

Remark 5 *As blow-up can not be excluded nor can global existence be proved in general, it remains an open problem to rigorously prove that any solution eventually converges to a trivial steady-state as the numerical experiments suggest. The nonlocal nature of the equation also seem to prevent the use of an appropriate concept of viscosity solution to obtain global existence in the class of, maybe,* C^1*-solutions, which could only develop a singularity in the second derivatives if they could be proven to develop a singularity at all.*

Proposition 6 *Let u_0 be a piecewise affine function vanishing at the end points with finitely many kinks located at interior points $x_j \in (0,1)$, $j = 1, \dots, N$. Then u_0 is a stationary solution of* (2.20)*.*

PROOF. On the given assumptions

$$-\partial_{xx} u_0 = \sum_{j=1}^{N} \alpha_j \delta_{x_j}$$

for appropriate $\alpha_j \in \mathbb{R}$. Since it is assumed that $u_0(x) = 0\,,\ x = 0, 1$, u_0 is a $\mathrm{H}^1_p(0,1)$-weak solution (see [28]) of

$$\begin{cases} \partial_{xx} u = \sum_{j=1}^{N} \alpha_j \delta_{x_j} & \text{in } (0,1)\,, \\ u = 0 & \text{on } \{0,1\}\,, \end{cases}$$

Furthermore

$$-\partial_{xx} u_0 \in \mathrm{H}^{-\rho}_p(0,1) = \left[\mathring{\mathrm{H}}^{\rho}_{p'}(0,1)\right]'$$

for $\rho > 1/p'$, where p' is the dual exponent to $p \in (1,\infty)$ since

$$\delta_{x_j} \in \Big[\overset{\circ}{\mathrm{H}}{}^{\rho}_{p'}(0,1)\Big]' \text{ by virtue of } \overset{\circ}{\mathrm{H}}{}^{\rho}_{p'}(0,1) \hookrightarrow \mathrm{C}([0,1])\,.$$

It is known [28] that one can associate an operator A_p to the Dirichlet form

$$a(u,v) = \int_0^1 u_x v_x\, dx\ ,\ u,v \in \overset{\circ}{\mathrm{H}}{}^1_p(0,1) \times \overset{\circ}{\mathrm{H}}{}^1_{p'}(0,1)\,,$$

which satisfies

$$A_p \in \mathcal{L}_{is}\Big(\overset{\circ}{\mathrm{H}}{}^1_p(0,1), \mathrm{H}^{-1}_p(0,1)\Big)\,.$$

It is also known [28, Theorem 8.5] that the $\mathrm{H}^{2-\rho}_{p,0} := \mathrm{H}^{2-\rho}_p(0,1) \cap \overset{\circ}{\mathrm{H}}{}^1_p(0,1)$-realization $A_{2-\rho,p}$ of A_p satisfies

$$A_{2-\rho,p} \in \mathcal{L}_{is}\Big(\mathrm{H}^{2-\rho}_{p,0}, \mathrm{H}^{-\rho}_p(0,1)\Big)\,.$$

It follows that $u_0 \in \mathrm{H}^{2-\rho}_{p,0}(0,1)$. Thus, the boundedness of the imaginary powers of the Dirichlet Laplacian implies that the domains of its fractional powers $A^{1-\varepsilon}$ are characterized by the corresponding complex interpolation spaces $H^{2-2\varepsilon}_{p,0}$. By means of the reiteration theorem [28, Theorem 7.2] it then follows that

$$A^{1-\varepsilon}_p u_0 \in \mathrm{H}^{2\varepsilon-\rho}_p(0,1)\,.$$

If $2\varepsilon > \rho > 1/p'$, which can always be achieved since p' can be chosen arbitrarily large, it is finally obtained that

$$A^{1-\varepsilon}_p u_0 = A^{-\varepsilon}_p \sum_{j=1}^N \alpha_j \delta_{x_j} \in \mathrm{L}_p(0,1)$$

for a $p > 1$ sufficiently close to 1. Notice that $A^{1-\varepsilon}_p u_0$ only has singularities where $\partial_{xx} u_0$ does as well (for ε small), which are, however, integrable. It follows that

$$a_\varepsilon(u_0) = \frac{1}{1 + |A^{1-\varepsilon}_p u_0|^2} \in \mathrm{C}\big([0,1]\big)$$

and that $a_\varepsilon(u_0)$ vanishes exactly at the points where the singularities are located. In summary

$$a_\varepsilon(u_0) \sum_{j=1}^N \alpha_j \delta_{x_j} = 0$$

and u_0 is indeed a stationary solution of the problem.

3 Numerical Experiments

Next a numerical discretization of (2.20) is derived and used to perform numerical experiments meant to illustrate the interesting dynamical behavior it engenders and some enhancements over the Perona-Malik equation which are, however, more thoroughly investigated in [1]. The operator A_D is discretized spectrally by means of the discrete sine transform S_n

$$A_n = S_n^{-1}\Lambda_n S_n \tag{3.1}$$

where $n = 2^m$ denotes the number of grid points used and

$$\begin{cases} D_n = \pi^2 \operatorname{diag}(1, 4, \dots, (n+1)^2)\,, \\ S_n = \sin(k\pi \frac{j}{n})\,,\ 1 \le k \le n+1\,,\ 0 \le j \le n\,. \end{cases}$$

The time variable is discretized by a forward semi-implicit Euler scheme so that

$$u^{k+1} = \Big[\mathrm{id}_n - \frac{h_t}{1 + (A_n^{1-\varepsilon} u^k)^2} A_n\Big]^{-1} u^k \tag{3.2}$$

where u^k is the spatial n-vector at time $k\,h_t$ for the time step $h_t > 0$. Observe that, setting $\varepsilon = 0$, a discretization of a “once integrated” version of the classical Perona-Malik equation is recovered.
In all of the following experiments $c = 0.025$, $m = 8$ and $h_t = 0.05$. First the evolution of the function

$$u_0(x) := 10\sin(2\pi x)\,,\ x \in [0, 1]\,.$$

is considered. Figure 1 depicts the first derivative of the solution at various times for $\varepsilon = 0$ (on the left) and $\varepsilon = 0.1$ (on the right). For $\varepsilon = 0$ one of the known shortcomings of the classical Perona-Malik model, “stair-casing”, is apparent. For $\varepsilon = 0.1$, stair-casing is avoided but the solution still tends to assume a piecewise constant shape. This example also hints at the de-blurring qualities of (2.20). Generic solutions to smooth initial data feel the presence of the nontrivial steady-states and tend to flow in their vicinity at all times. This is exemplified in Figure 2 where various stages of the evolution of a smooth solution with initial condition

$$u_0(x) = 10\sin(6\pi x) + 2\sin(4\pi x)\,,\ x \in [0, 1]\,.$$

are shown. This numerical experiment seems to indicate that the solution might exist globally in some sense and that the non-trivial equilibria are responsible for its transient behavior while diffusion takes over in the long run, eventually driving the solution to the trivial steady-state. Clearly this convergence might be due to numerical diffusion. However, conservation relation (2.21) provides an indirect way to test the numerics. It can be tracked during

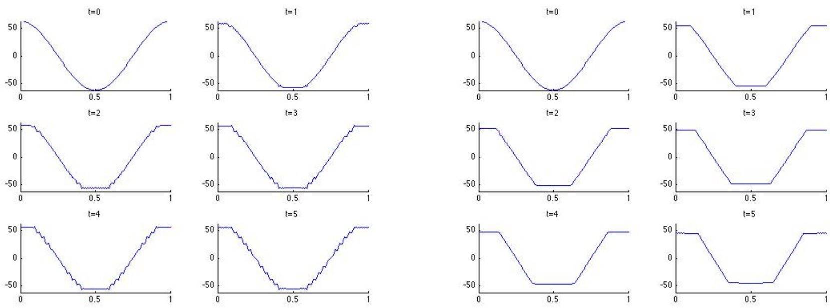

Fig. 1. The gradient of the solution for $\varepsilon = 0$, with stair-casing, and for $\varepsilon = 0.1$, without this artifact.

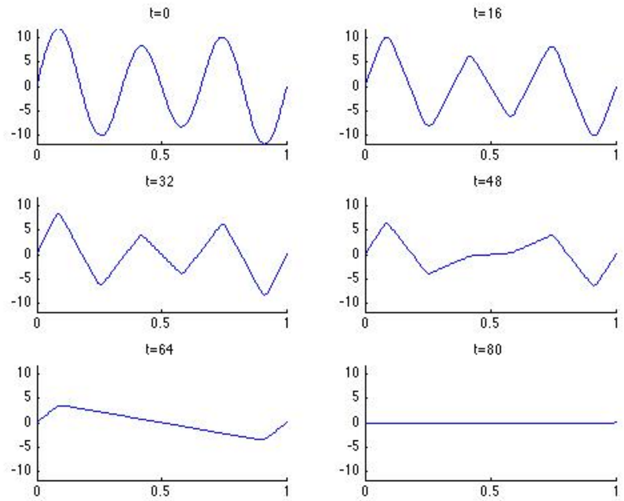

Fig. 2. Tendency to evolve towards or close to piece-wise affine functions.

the solution's evolution and, in the particular case considered, the relative numerical deviation observed between the left and right-hand-side of it amounts to a mere 0.42% for the fully converged solution. Scheme (3.2) also preserves the piecewise affine structure of initial values for large integration times. The solution with initial condition

$$u_0(x) = 20 - |40\,x - 20|\,,\ x \in (0,1)\,,$$

is computed up to time $t = 8$. The solutions are plotted in Figure 3. The colors blue, black, magenta and red correspond to the solution with $\varepsilon = 0, 0.1, 0.2, 0.3$, respectively. In spite of the fact that u_0 is a steady-state for any choice of $\varepsilon \in (0, \frac{1}{2})$, dissipation is stronger for larger ε as the relative strength of the non-linearity decreases. The initial value and the solutions to $\varepsilon = 0, 0.1$ are indistinguishable in the plot. Continuous piecewise linear functions are not steady-states for the original Perona-Malik equation since the non-linearity is not well-defined for such functions. In spite of this its numerical counterpart delivers results comparable to those for small positive ε. Figure

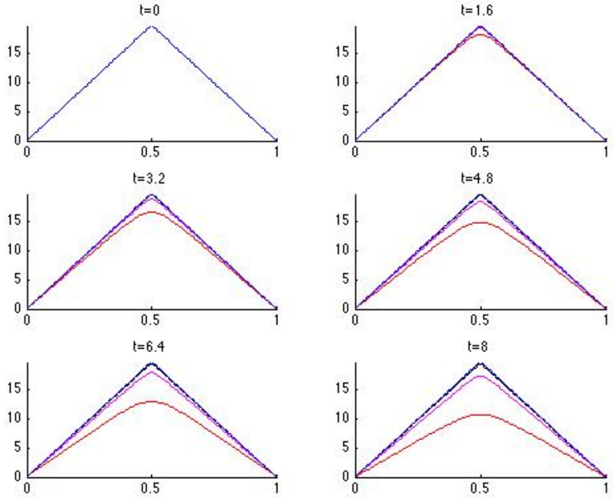

Fig. 3. Behavior close to formally stationary solutions.

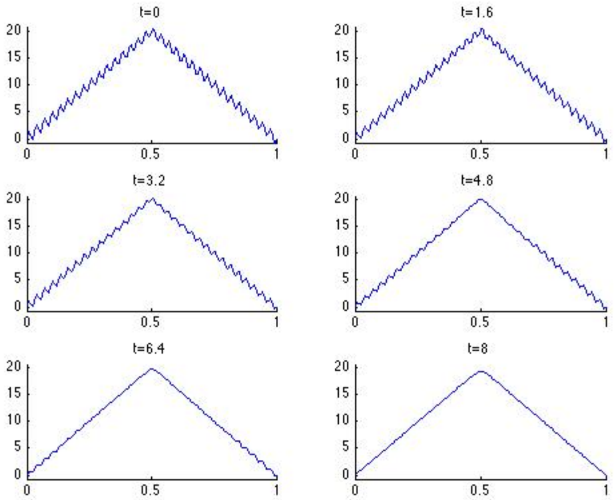

Fig. 4. The effect of (2.20) on high-frequency low-contrast oscillations. The initial condition tends to evolve back to the underlying piecewise affine function.

4 shows clearly an additional benefit of the new equation (2.20) as compared to the original Perona-Malik equation. Depicted is the evolution of

$$u_0(x) = 20 - |40\,x - 20| + \sin(64\pi x)\,,\ x \in (0,1)\,,$$

up to time $t = 8$ with $\varepsilon = 0.2$. The new equation can manifestly differentiate between high low-contrast gradients and high-contrast gradients, which are remarkably well preserved. Observe that the initial oscillatory condition would be left visually and virtually unchanged by the original Perona-Malik equation for the same and longer time ranges. The qualities predicted in the previous one-dimensional experiments are also observed in two dimensional experiments

Fig. 5. The denoising effect obtained by means of (1.1) with $\varepsilon = 0.6$. The initial condition is Lenna's image corrupted with about 15% salt and pepper noise.

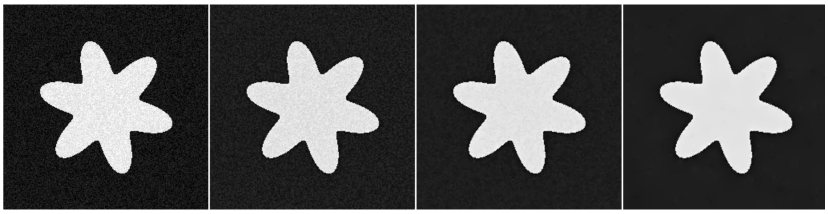

Fig. 6. The denoising effect obtained with (1.1) for $\varepsilon = 0.4$. The initial condition is a 15%-noise-corrupted characteristic function

based on (1.1). Figures 5 and 6 show the evolution of a noisy test image every pixel of which has been corrupted by about 15% noise in the gray-scale. For other tests and details of the two-dimensional implementation we refer to [1].

4 Conclusions

A new nonlinear nonlocal diffusion has been proposed and analyzed which gives rise to an interesting dynamical behavior. In a one dimensional setting it can be related by strength reduction in the nonlinearity obtained by means of a non-local term to the celebrated Perona-Malik equation. Well-posedness of the new equation and its dynamical behavior are its most striking analytical features. From the practical point of view, the new equation delivers enhanced benefits as compared to Perona-Malik and suppresses known shortcomings associated to it. A more extensive investigation of this issue is contained in [1] but the benefits of (1.1) clearly owe to the existence of the class of meta-stable non-trivial equilibria driving the evolution of generic solutions pointed out in this paper.

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