A New Nonlocal Nonlinear Diffusion of Image Processing

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Abstract

A novel nonlocal nonlinear diffusion is analyzed which has proven useful as a denoising tool in image processing. The equation can be viewed as a new paradigm for the regularization of the well-known Perona-Malik equation. The regularization is implemented via nonlinearity intensity reduction through fractional derivatives. Well-posedness in the weak setting is established. Global existence and convergence to the average holds in the purely diffusive limit whereas an interesting dynamic behavior is engendered by the presence of non trivial equilibria as the intensity of the nonlinearity is increased and comes close to the one of Perona-Malik.

Key words: Nonlinear nonlocal diffusion, well-posedness, noise reduction, Perona-Malik equation, non trivial equilibria.

1 Introduction

Recently two novel nonlinear diffusions have been proposed [1] for the purpose of denoising in image processing. One of the proposed models has been analyzed in [2], while the other is considered here. It reads

$$\begin{cases} u_t - \operatorname{div}\Big(\frac{1}{1+|\nabla^{1-\varepsilon}u|^2}\nabla u\Big) = 0\,, & \text{in } \Omega \\ u & \text{periodic}\,, \end{cases} \tag{1.1}$$

where $\Omega = [0,1]^n \subset \mathbb{R}^n$ for $n \geq 1$ and $\varepsilon \in (0,1)$. Equation (1.1) can be considered as a novel regularization of the celebrated Perona-Malik equation of

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image processing which generated a considerable amount of interest in the last twenty years. It is obtained from (1.1) by setting $\varepsilon = 0$. The Perona-Malik equation has been shown to be ill-posed in [3]. It can, however, be successfully implemented to give a viable denoising tool for image processing with remarkable preservation of sharp edges. This discrepancy between its mathematical properties and its practical stability and efficacy has been termed "Perona-Malik paradox" by Kichenassamy. It is one of the reasons why the mathematical community [3–12] has shown an unfading interest in the equations since they were first introduced in [13]. Many a researcher have proposed regularized and/or relaxed models ([14–20]) which could be shown to be at least locally well-posed and which could preserve the cherished features of Perona-Malik to some extent. All models, however, tend to introduce a certain amount of blurring and thus compromise one of the salient qualities of the Perona-Malik model. The latter can be viewed as the gradient flow generated by the non-convex functional

$$E(u) = \frac{1}{2} \int_\Omega \log\Big(1 + |\nabla u|^2\Big)\, dx\,. \tag{1.2}$$

In the one dimensional setting piecewise constant functions seem to play an important role in the understanding of the dynamical behavior of numerical solutions of the Perona-Malik equation. Kichenassamy introduced an ad-hoc concept of stationary solution in [3] which make them stationary solutions of Perona-Malik. In [21] it is argued that they ought to be considered stationary points of the energy functional (1.2) if the $\nabla u = u_x$ is taken to be the absolutely continuous part of the first derivative. For a piecewise constant function one has

$$u_x = \sum_{j=1}^{n} \alpha_j \delta_{x_j}$$

for some $\alpha_j \in \mathbb{R}$ and $x_j \in (0,1)$ for $j = 1, \dots, n$ and thus it is impossible to make mathematical sense of the nonlinearity applied to it. Inspired by the theory of laminates, in [5] and [6] the authors are able to use the concept of Young-measure valued solutions to shed some light on the "staircasing effect" observed in practical implementations of Perona-Malik. Numerical solutions tend to become piecewise constants on small scales thus producing the characteristic visual effect of a staircase. This effect is, however, been observed to be highly sensitive on the mesh-size of any particular numerical implementation. In [6] the existence of infinitely many weak Young-measure valued solutions of a certain kind reflects the numerical unpredictability of the effect. Still in a variational vein, a recent paper by Bellettini and Fusco [11] makes use of De Giorgi's minimizing motions to give a sound mathematical model for the evolution of non-smooth initial data driven by an energy functional defined on a space of piecewise constant functions which they obtain via Γ-convergence from a functional with a higher order regularization term.

Taking a different perspective but with the goal of mitigating if not resolving the ill-posedness issue, a number of researchers have turned to considering semi-discrete models where the spatial variable is discretized. Taking special limits or analyzing the semi-discrete model in its own right, [22–24] have obtained results which agree with and therefore shed some light on numerical observations.

It should be stressed that all of the above results concerning Perona-Malik or related semi-discrete models were obtained in the one dimensional case.

The advantages of (1.1) are manifold. First it will be shown to be well-posed; second, piecewise constant functions or characteristic functions of smooth sets in higher dimensions can be shown to be equilibria for the evolution. Third, the combination of these features leads to well-behaved discretizations which preserve the most cherished properties of Perona-Malik while significantly alleviating its known short-comings such as staircasing, poor performance in flat regions and undesired preservation of small scale features due to noise. It should be observed that equation (1.1) does not have a variational structure. It, however, has a natural Lyapunov function given by

$$L(u) = \frac{1}{2}\int_\Omega |u|^2\,dx \tag{1.3}$$

since

$$\frac{d}{dt}L(u) = -\int_\Omega \frac{|\nabla u|^2}{1+|\nabla^{1-\varepsilon}u|^2}\,dx\,.$$

Thus, in a way, (1.1) demonstrates that non-convex functionals, or backward-forward diffusion, are not a necessary ingredient for non-trivial transient behavior in a purely diffusive equation with no reaction term.

Finally it is observed that there exist a considerable amount of literature on the general subject of forward-backward diffusions (see for instance [25–28]) but the available results do require additional hypotheses to be satisfied which preclude an application to the Perona-Malik model with the sole exception of the already mentioned [5,6].

2 Preliminaries

2.1 The Fractional Gradient

In order to make sense of the equation, the fractional derivative appearing in the diffusion coefficient needs to be defined. As periodic boundary conditions are considered here, this will be done via Fourier transform

$$\mathcal{F}(u)(k) = \hat{u}(k) = \int_\Omega e^{-2\pi i k\cdot x} u(x)\, dx\,,\ k \in \mathbb{Z}^n\,.$$

One clearly has that

$$\partial_j = \mathcal{F}^{-1}\operatorname{diag}[(-2\pi i k_j)_{k\in\mathbb{Z}^n}]\mathcal{F}\,, j = 1,\dots,n\,,$$

where $\operatorname{diag}[(m_l)_{l\in\mathbb{Z}^n}]$ denotes the multiplication operator with entries given by the sequence $(m_l)_{l\in\mathbb{Z}^n}$ and k_j is identified with $(0,\dots,0,\underbrace{k_j}_{\text{j-th coordinate}},0,\dots,0)$.

Then the fractional gradient

$$\nabla^{1-\varepsilon} = (\partial_1^{1-\varepsilon},\dots,\partial_n^{1-\varepsilon})$$

is defined via

$$\partial_j^{1-\varepsilon} = \mathcal{F}^{-1}\operatorname{diag}\Big[(2\pi|k_j|)^{1-\varepsilon}e^{-i\pi(1-\varepsilon)\operatorname{sign}(k_j)}\Big]\mathcal{F}\,. \tag{2.1}$$

While other possible definitions would work as well, the chosen one has the advantage of making it straightforward to characterize the mapping behavior of the fractional derivatives between Bessel potential spaces. It also make it particularly easy to discretize fractional derivatives in Fourier space by means of the Fast Fourier transform.

Remark 1 *Observe that, for $\varepsilon = 0$, the standard definition of the gradient is recovered. As for $\varepsilon = 1$, the definition*

$$\partial_j^0 u = u - \bar{u}\,,\ u \in \mathrm{L}_2(\Omega)\,,\ \textit{where } \bar{u} = \frac{1}{|\Omega|}\int_\Omega u\, dx\,,$$

is chosen. This is natural as the range of the fractional gradient for $\varepsilon < 1$ is always contained in the subspace of mean zero functions.

2.2 Maximal Regularity

In order to show that (1.1) is well-posed, its quasilinear structure will be exploited. Maximal or optimal regularity results of L_p-type are particularly well suited for this purpose. Given $T > 0$ and a Banach space E_0, consider the abstract Cauchy problem

$$\begin{cases} \dot{u} + A(t)u = F(t)\,, & t \in (0,T] =: J_T\,,\\ u(0) = u_0\,, \end{cases} \tag{2.2}$$

for an unknown function $u : J_T \to E_0$ and for

$$A \in \mathrm{L}_\infty\Big(J_T, \mathcal{L}(E_1, E_0)\Big) \text{ and } F \in \mathrm{L}_p(J_T, E_0)\,,$$

such that

$$\mathrm{dom}\Big(A(t)\Big) \doteq E_1 \overset{d}{\hookrightarrow} E_0\,,\ t \in J_T\,.$$

The Cauchy problem (2.2) is said to enjoy L_p-maximal regularity iff there exists a unique solution

$$u \in \mathbb{E}_{1,p}(J_T) := \mathrm{L}_p(J_T, E_1) \cap \mathrm{H}^1_p(J_T, E_0)\,,$$

for each $F \in \mathbb{E}_{0,p}(J_T) := \mathrm{L}_p(J_T, E_0)$ and $u_0 = 0$. In other words, maximal regularity holds if both functions on the left-hand-side $[t \mapsto \dot{u}(t)]$ and $[t \mapsto A(t)u(t)]$ inherit the regularity of the right-hand-side F. It is known that, in this case, $A(t)$ is the generator of an analytic semigroup. Using interpolation theory and the characterization of interpolation spaces by traces, it is well-known that the homogeneous initial condition $u_0 = 0$ can be replaced by

$$u_0 \in (E_0, E_1)_{1-\frac{1}{p},p} =: E_{1-\frac{1}{p}}\,,$$

where $(\cdot,\cdot)_{\theta,p}$ denotes the standard real interpolation functor. Thus maximal regularity is equivalent to

$$\partial_t + A(t) \in \mathcal{L}_{is}\Big(\mathbb{E}_{1,p}(J_T), \mathbb{E}_{0,p}(J_T) \times E_{1-\frac{1}{p}}\Big)\,.$$

The collection of operator families $A \in \mathrm{L}_\infty\Big(J_T, \mathcal{L}(E_1, E_0)\Big)$ satisfying the maximal regularity condition is denoted by $\mathcal{MR}_p(J_T)$ and is endowed with the topology induced by $\mathrm{L}_\infty\Big(J_T, \mathcal{L}(E_1, E_0)\Big)$. Maximal regularity is very useful to deal with quasilinear abstract Cauchy problems such as

$$\begin{cases} \dot{u} + A(u)u = F(u)\,, & t \in J_T\,, \\ u(0) = u_0\,. \end{cases} \tag{2.3}$$

Assume that

$$A \in \mathrm{C}^{1-}_b\Big(\mathbb{E}_{1,p}(J_T), \mathcal{MR}_p(J_T)\Big) \tag{2.4}$$

$$F \in \mathrm{C}^{1-}_b\Big(\mathbb{E}_{1,p}(J_T), \mathbb{E}_{0,p,q}(J_T) \times E_{1-\frac{1}{p}}\Big)\,,\ q > p\,, \tag{2.5}$$

where the superscript indicates Lipschitz continuity and the subscript boundedness and uniform Lipschitz continuity on bounded subsets. Also

$$\mathbb{E}_{0,p,q}(J_T) := \Big\{F : \mathbb{E}_{1,p}(J_T) \to \mathbb{E}_{0,q}(J_T) \,|\, F - F(0) \in \mathrm{C}^{1-}_b\Big(\mathbb{E}_{1,p}, \mathbb{E}_{0,p}(J_T)\Big)\Big\}\,.$$

The following theorem is proved in [29].

Theorem 2 *Let* (2.4)-(2.5) *be satisfied. Then* (2.3) *has a unique maximal solution* $u : J_{T_{max}} \to E_0$ *for* $T_{max} > 0$ *with*

$$u \in \mathbb{E}_{1,p}(J_S) \text{ for } S < T_{max}\,.$$

Moreover $T_{max} = T$*, that is, the solution exists globally in time, if* $u \in \mathbb{E}_{1,p}(J_{T_{max}})$ *or if* $A(u) \in \mathcal{MR}_p(J_{T_{max}})$.

2.3 Weak L_p*-formulation*

In order to be able to use maximal regularity to solve of (1.1) an appropriate weak L_p-formulation is needed. Denote by

$$\mathrm{H}^1_{p,\pi}(\Omega)$$

the space of periodic H^1_p-functions and consider the bilinear form

$$a[u](v,w) := \int_\Omega \frac{1}{1+|\nabla^{1-\varepsilon}u|^2}\nabla v \nabla w\, dx\,, \qquad v \in \mathrm{H}^1_{p,\pi}(\Omega)\,,\ w \in \mathrm{H}^1_{p',\pi}(\Omega)\,, \tag{2.6}$$

for any given $u \in \mathrm{H}^1_{p,\pi}(\Omega)$. Classical embedding theorems give

$$\mathrm{H}^\delta_{p,\pi}(\Omega) \hookrightarrow \mathrm{C}^{\delta-\frac{n}{p}}_\pi(\overline{\Omega})\,, \tag{2.7}$$

as soon as $\delta - \frac{n}{p} > 0$. Thus, choosing $p >> 1$ large enough, it can be made sure that $a[u]$ is well-defined for $u \in \mathrm{H}^1_{p,\pi}(\Omega)$ as

$$\nabla^{1-\varepsilon}u \in \mathrm{C}^\rho_\pi(\overline{\Omega})^n\,,\ \text{for } \epsilon - \frac{n}{p} \geq \rho > 0\,.$$

The form $a[u]$ naturally induces an operator from

$$A(u) \in \mathcal{L}\Big(\mathrm{H}^1_{p,\pi}(\Omega), \mathrm{H}^{-1}_{p,\pi}(\Omega)\Big)\,,$$

where $\mathrm{H}^{-1}_{p,\pi}(\Omega) = \mathrm{H}^1_{p',\pi}(\Omega)'$. It follows from the general results of [30] that $-A(u)$ generates an analytic semigroup on $\mathrm{H}^{-1}_{p,\pi}(\Omega)$ with domain $\mathrm{H}^1_{p,\pi}(\Omega)$ for any choice of $u \in \mathrm{H}^1_{p,\pi}(\Omega)$. Even more is actually true. It follows in fact from [31,32] that

$$A(u) \in \mathcal{MR}_p(J_T) \text{ for any } T > 0\,. \tag{2.8}$$

Problem (1.1) can be reformulated weakly as

$$\begin{cases} \langle \dot{u}, \varphi\rangle - a[u](u,\varphi) = 0\,,\ \varphi \in \mathrm{H}^1_{p',\pi}(\Omega)\,,\ \text{a.e. in } t \in J_T\,, \\ u(0) = u_0\,, \end{cases} \tag{2.9}$$

which amounts to the quasilinear abstract Cauchy problem

$$\begin{cases} \dot{u} + A(u)u = 0\,, \\ u(0) = u_0\,, \end{cases} \tag{2.10}$$

in the space $\mathrm{H}^{-1}_{p,\pi}(\Omega)$.

3 Well-posedness

3.1 Local Existence

If A satisfies (2.4) it is possible to use Theorem 2 to obtain a local existence theorem. For that purpose, assume that

$$u_0 \in \Big(\mathrm{H}^{-1}_{p,\pi}(\Omega), \mathrm{H}^{1}_{p,\pi}(\Omega)\Big)_{1-\frac{1}{p},p} = \mathrm{H}^{1-\frac{2}{p}}_{p,\pi}(\Omega) = E_{1-\frac{1}{p}}\,, \tag{3.1}$$

if $E_0 := \mathrm{H}^{-1}_{p,\pi}(\Omega)$ and $E_1 := \mathrm{H}^{1}_{p,\pi}(\Omega)$. Using the embedding (2.7) and known results concerning the regularity of Nemitskii operators on spaces of Hölder continuous functions [33] it is now possible to show that

$$A \in \mathrm{C}^{1-}_b\Big(\mathbb{E}_{1,p}(J_T), \mathcal{MR}_p(J_T)\Big)$$

in the sense of (2.4).

Theorem 3 *Problem* (1.1) *possesses a unique maximal weak solution in the sense of* (2.9) *which satisfies*

$$u \in \mathrm{L}_p\Big(J_{max}, \mathrm{H}^1_{p,\pi}(\Omega)\Big) \cap \mathrm{H}^1_p\Big(\dot{J}_{T_{max}}, \mathrm{H}^{-1}_{p,\pi}(\Omega)\Big)$$

for any given $u_0 \in \mathrm{H}^{1-\frac{2}{p}}_{p,\pi}(\Omega)$. *The solution is global if it can be shown that*

$$u \in \mathrm{H}^1_p\Big(J_{max}, \mathrm{H}^{-1}_{p,\pi}(\Omega)\Big) \text{ or that } A(u) \in \mathcal{MR}_p(J_{T_{max}})\,.$$

PROOF. Using the results of [33] on Nemitskii operators on spaces of Hölder continuous functions and trivial properties of the function $[s \mapsto \frac{1}{1+s^2}]$, it follows from the considerations around (2.7) that

$$[u \mapsto \frac{1}{1+|\nabla^{1-\varepsilon}u|^2}]\,,\ \mathrm{C}^\rho_\pi(\overline{\Omega}) \to \mathrm{C}^\rho_\pi(\overline{\Omega}) \tag{3.2}$$

is a Lipschitz map which is bounded and uniformly Lipschitz on bounded subsets. In fact, in order to apply the results of [33] it suffices to check that

$$[s \mapsto \frac{1}{1+s^2}] \in \mathrm{BUC}^2(\mathbb{R}, \mathbb{R}) \, .$$

The fractional gradient operator is better considered as acting between Bessel potential spaces. In that case its mapping properties can be directly obtained by characterization of such spaces via decay properties of Fourier coefficients and the explicit knowledge of its symbol (2.1). The mapping result is then obtained by factoring the Nemitskii operator via the embedding (2.7). Next it follows that

$$[u \mapsto \frac{1}{1+|\nabla^{1-\varepsilon} u|^2}] \, , \; \mathbb{E}_{1,p}(J_T) \to \mathrm{C}\Big(J_T, \mathrm{C}^\rho(\overline{\Omega})\Big)$$

since

$$\mathbb{E}_{1,p}(J_T) \hookrightarrow \mathrm{C}(J_T, E_{1-\frac{1}{p}}) \hookrightarrow \mathrm{C}\Big(J_T, \mathrm{C}^\rho(\overline{\Omega})\Big)$$

as follows from (2.7) by choosing p large enough and ρ small enough. An analytic dependence result for differential operators (in weak form) on their coefficients [30, Theorem 8.5] combined with [34, Theorem 7.1] and (2.8) gives the desired mapping property (2.4). It is recalled that [34, Theorem 7.1] states that

$$\mathrm{C}\Big(J_T, \mathcal{MR}_p\Big) \subset \mathcal{MR}_p(J_T) \, ,$$

where $\mathcal{MR}_p$ denotes the subspace of constant (in time) operators with maximal regularity. In other words, continuous dependence in time is sufficient for maximal regularity if the latter property is enjoyed by each single operator in a time dependent family.

3.2 Global Existence

The intensity of the nonlinearity can be tuned by the choice of $\varepsilon > 0$. Choosing $\varepsilon = 1$ leads to the nonlinear diffusion equation

$$u_t - \mathrm{div}\Big(\frac{1}{1+n^2(u-\bar{u})^2} \nabla u\Big) = 0 \tag{3.3}$$

At the other end, for $\varepsilon = 0$, one finds the Perona-Malik equation which is ill-posed. It will be proved in this section that the weak solution of (3.3) exists globally and converges to its average, thus exhibiting a standard diffusive behavior. Since global existence cannot be expected for $\varepsilon = 1$, it is an open and interesting question to determine the critical value of ε beyond which global existence is lost. Numerical experiments do suggest that standard diffusive behavior and global existence might be valid up to $\varepsilon = 1/2$, while for $\varepsilon > 1/2$, gradient blowup seems possible.

Equation (3.3) will be considered with the nonlinearity constant $n = 1$ (but arbitrary dimension) since the analysis is not affected by its value. It is plain that the average $\bar{u}_0$ is preserved during the evolution regardless of $\varepsilon \in [0, 1)$. A classical argument also gives that

$$\inf_{x\in\Omega} u_0(x) \leq u(t,x) \leq \sup_{x\in\Omega} u_0(x) \text{ a.e in } \Omega\,. \tag{3.4}$$

Next classical results can be used to obtain global existence.

Proposition 4 *Let u be the local weak solution of* (3.3) *on its maximal interval of existence* $[0, T_{max})$. *Then* $T_{max} = \infty$, *that is, the solution is global and it converges to its mean value.*

PROOF. Classical results concerning parabolic regularity [35] combined with (3.4) make sure that the solution u is locally Hölder continuous with respect to all variables. Since periodic boundary conditions are imposed here and the initial datum is Hölder continuous, this regularity is global. Thus for some parameter $\rho > 0$, it holds that

$$u \in \mathrm{C}^{\rho/2}\Big([0,T], \mathrm{C}^{\rho}_{\pi}(\overline{\Omega})\Big)\,.$$

It follows from the considerations at the end of Section 3.1 that

$$[t \mapsto A(u(t,\cdot))] \in \mathrm{C}^{\rho/2}\Big([0,T], \mathcal{MR}_p\Big) \subset \mathcal{MR}_p(J_T)\,.$$

Theorem 3 then shows that the weak solution of (1.1) is global. Convergence towards its mean follows from

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (u-\bar{u})^2\,dx \leq -\int_\Omega \frac{|\nabla(u-\bar{u})|^2}{1+(u-\bar{u})^2}\,dx \\ \leq -\frac{1}{1+4\|u\|_\infty^2}\int_\Omega |\nabla(u-\bar{u})|^2\,dx \\ \leq -\frac{c}{1+4\|u\|_\infty^2}\int_\Omega |(u-\bar{u})|^2\,dx\,,\end{aligned}$$

where Poincaré's inequality was used in the last inequality.

4 Stationary Solutions

Numerical experiments show that characteristic functions play an important role in the evolution engendered by (1.1), at least for ε close to zero. In fact a more standard diffusive behavior is observed for ε closer to 1. In this section it is shown that characteristic functions of sets with smooth boundaries are indeed stationary solutions of (1.1). The numerical experiments performed in the next section do suggest that they at least are meta-stable, in that they attract solutions over a large time interval before diffusion takes over, eventually driving them to a trivial steady-state. As blow-up can not be ruled out, it is not clear whether the long time behavior of solutions is really well represented by their numerical approximations. Similar behavior has, however, been reported for the Perona-Malik equation ([3]) and for some of its discrete counterparts ([22,11]). Kichenassamy, in particular, introduces, in a one-dimensional setting an ad-hoc definition of solution which allows for piecewise constant functions to be equilibria for the problem. The nonlinearity is, however, not defined in that case as a nonlinear function of a distribution. The use of the fractional gradient considered here allows one to overcome that difficulty. In fact, when applied to a characteristic function of a smooth set it leads to a regular distribution which is singular along the boundary of the set but globally integrable, that is a regular distribution. The nonlinearity can thus be applied to it and delivers a function which vanishes on the boundary. As the gradient of the characteristic function is a vector measure supported on the boundary, multiplication can be justified and is seen to vanish.

Lemma 5 *Let $S \subset \Omega$ be a subset of Ω with smooth boundary ∂S and consider its characteristic function χ_S. Then $|\nabla^{1-\varepsilon}\chi_S| \in \mathrm{L}_{1,\pi}(\Omega)$ and is smooth away from ∂S.*

PROOF. Observe first that, for any $p \in (1,\infty)$,

$$\partial_j^{1-\varepsilon}\chi_S \in \mathrm{L}_{p,\pi}(\Omega) \text{ for } j=1,\dots,n \text{ iff } (-\triangle)^{\frac{1-\varepsilon}{2}}\chi_S \in \mathrm{L}_{p,\pi}\,,$$

provided that

$$(-\triangle)^{-\frac{1-\varepsilon}{2}}\partial_j^{1-\varepsilon} \in \mathcal{L}_{is}\Big(\mathrm{L}_{p,\pi,0}(\Omega)\Big)\,,\ j=1,\dots,n\,, \tag{4.1}$$

where the additional subscript indicates the closed subspace of mean zero functions in $\mathrm{L}_{p,\pi}(\Omega)$. This is justified since clearly

$$\nabla^{1-\varepsilon}\chi_S = \nabla^{1-\varepsilon}\Big(\chi_S - \overline{\chi_S}\Big)\,,$$

according to definition (2.1). Property (4.1) follows from the explicit knowledge of the operator symbols, properties of polynomials and classical Fourier multiplier theorems [36]. Next by similar arguments it follows that

$$(-\triangle)^{\frac{1}{2}}\chi_S \in \mathrm{H}^{-s}_{p,\pi}(\Omega) \text{ iff } |\partial_j\chi_S| \in \mathrm{H}^{-s}_{p,\pi}(\Omega) \text{ for } j=1,\dots,n\,.$$

It is known [37] that $\partial_j\chi_S$ is a measure supported on the boundary ∂S and therefore $\partial_j\chi_S \in \mathrm{H}^{-s}_{p,\pi}(\Omega)$ follows for instance from

$$\mathrm{H}^{s}_{p',\pi}(\Omega) \hookrightarrow \mathrm{C}(\overline{\Omega})\,,$$

which is valid provided p is chosen sufficiently close to 1. Now

$$(-\triangle)^{\frac{1-\varepsilon}{2}}\chi_S \in \mathrm{H}^{\varepsilon-s}_{p,\pi}(\Omega) \hookrightarrow \mathrm{L}_{p,\pi}(\Omega)$$

as soon as $0 < s < \varepsilon$. Finally $\partial_j^{1-\varepsilon}$ is a pseudo differential operator, and, as such, can be represented as convolution with a generalized kernel K_j which is smooth off of its diagonal, that is,

$$\partial_j^{1-\varepsilon} u(x) = \int_\Omega K_j(x-\tilde{x})u(\tilde{x})\,dx\,.$$

It follows that if $u = \chi_S$ and $x \in \Omega \setminus \partial S$, then u is smooth and constant in a neighborhood of x. Thus, in a (possibly smaller) neighborhood $\partial_j^{1-\epsilon}u$ can be computed by the above formula where integration only occurs away from the singularity of the kernel (notice that, since $\partial_j^{1-\varepsilon}u = \partial_j^{1-\varepsilon}(u-c)$ for arbitrary c, it can always be assumed that χ_S is zero in the neighborhood considered). It is therefore smooth in that neighborhood.

Proposition 6 *Characteristic functions χ_S of sets with smooth boundary ∂S are stationary solutions for* (1.1)*.*

PROOF. By the preceding lemma it follows that $\frac{1}{1+|\nabla^{1-\varepsilon}\chi_S|^2}$ is a continuous function which vanishes on ∂S. As $\nabla\chi_S$ is a vector measure supported on ∂S, their product $\frac{1}{1+|\nabla^{1-\varepsilon}\chi_S|^2}\nabla\chi_S$ is well defined and vanishes as desired.

5 Some Numerical Experiments

A few numerical simulations will be shown here which highlight the role of the class of nontrivial equilibria of the previous section. They appear to determine the behavior of solutions to (1.1) for a long time before diffusion can take over and drive any solution to its mean. Since blowup can not be excluded so far in general, there is, however, no guarantee that convergence to a trivial steady-state is not merely a numerical occurrence. Figure 1 depicts the typical evolution of a smooth initial datum. Figure 2 demonstrates the ability of (1.1) to differentiate between high frequency global features from local ones. The evolution of a simple characteristic function polluted with noise is shown. Finally an example of a long time evolution in the two dimensional case is shown in Figure 3. The interested reader is referred to [1] for more exhaustive experiments extolling the main practical qualities of (1.1).

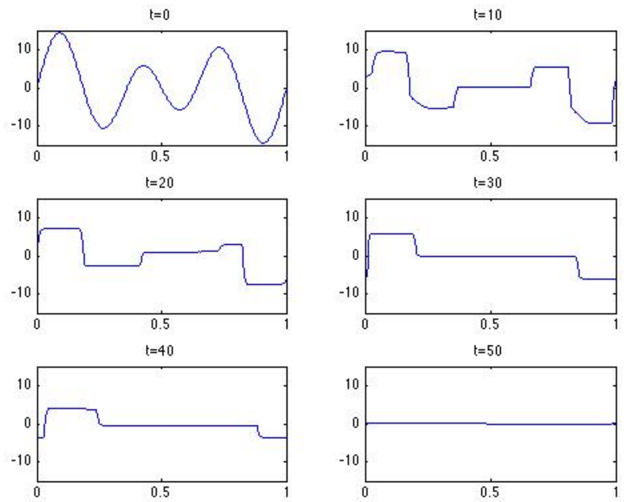

Fig. 1. Characteristic evolution of a smooth initial datum.

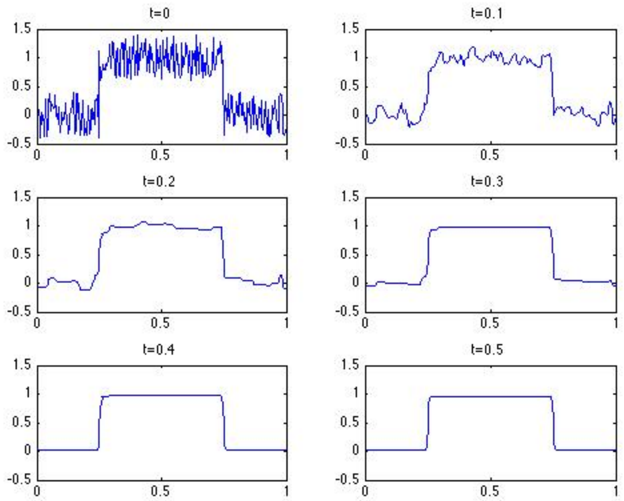

Fig. 2. Evolution of a characteristic function with additive noise.

6 Conclusions

A novel nonlinear nonlocal diffusion is analyzed which has recently been proposed as a denoising tool in image processing. It can be obtained by the classical Perona-Malik equation by a very mild regularization implemented by using fractional derivatives. It is shown to be well-posed and to possess characteristic functions of sets with smooth boundary as equilibria. Their presence engenders a non trivial dynamic behavior whereby a solution typically tends its average in the long run but first approaches a piecewise constant function and always follows a path close to such functions. The novel equations also

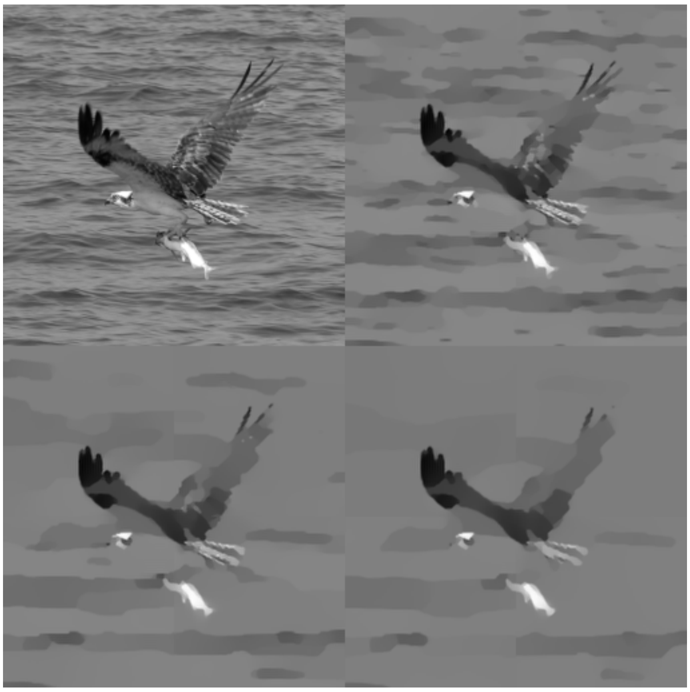

Fig. 3. Evolution of a two-dimensional image.

shows that a well-posed (forward) pure diffusion (no reaction term) can give rise to nontrivial dynamic behavior. This is to be contrasted with the common perception that more traditional anisotropic diffusions like Perona-Malik owe their non trivial dynamic behavior to their forward-backward nature. The new equation generates behaviors that interpolate between pure diffusion ($\varepsilon = 1$) and Perona-Malik's ill-posedness ($\varepsilon = 0$). It is an interesting open question to investigate the existence of a critical parameter value which separates the standard diffusive behavior from the one described in this paper.

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