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Optimal regularity for a class of singular abstract parabolic equations

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Abstract

A general class of singular abstract Cauchy problems is considered which naturally arises in applications to certain free boundary problems. Existence of an associated evolution operator characterizing its solutions is established and is subsequently used to derive optimal regularity results. The latter are well known to be important basic tools needed to deal with corresponding nonlinear Cauchy problems such as those associated to free boundary problems.

Keywords: Singular parabolic initial boundary value problems; Optimal regularity; Evolution operator

1. Introduction

In this paper a class of abstract Cauchy problems in a Banach space E_0

$$\dot{u} - A(t)u = f(t), \quad t > 0,$$

is considered where the family of operators A is allowed to have singular behavior in the origin. The paper will focus on the "parabolic case," that is, it will be assumed that the operators $A(t)$ generate analytic semigroups for fixed $t > 0$. It will, however, not be assumed that the latter be strongly continuous. Conditions on the family A (see (2.27)–(2.29)) are proposed which lead to the construction of an associated evolution operator U_A (cf. (3.6)). By means of it, maximal regularity results are obtained for the singular abstract Cauchy problem in the context of spaces of singularly Hölder continuous functions. A certain class of free boundary problems [18] with

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initial onset of a phase is the prime motivation for the study presented here. The paper is organized as follows. In the next section the problem is put into context and the basic tools needed in its analysis are presented. In Section 3 the evolution operator U_A is constructed and, in the following section, it is used to prove maximal regularity results for the singular abstract Cauchy problem. In final Section 5 an example is considered of a parabolic problem in a space–time wedge and the connection to free boundary problems is made more explicit.

2. Preliminaries and setting

Let E_0 be a Banach space. An unbounded operator

$$A : \mathrm{dom}(A) \subset E_0 \to E_0$$

is called *sectorial* if it satisfies

$$\text{(i)} \quad \overline{\mathrm{dom}(A)} = E_0, \qquad N(A) = \{0\}, \qquad \overline{R(A)} = E_0, \tag{2.1}$$

$$\text{(ii)} \quad (0,\infty) \subset \rho(A) \quad \text{and} \quad \left\| t(t-A)^{-1} \right\|_{\mathcal{L}(E_0)} \leqslant M, \quad t > 0,\ M > 0. \tag{2.2}$$

If A only satisfies (ii) is usually called *pseudo-sectorial*. For any given pseudo-sectorial operator A on E_0 there exists $\theta > 0$ such that

$$\rho(A) \supset \Sigma_\theta := \left\{ \lambda \in \mathrm{C} \setminus \{0\} \;\middle|\; \left|\arg(\lambda)\right| < \theta \right\} \quad \text{and} \quad \sup_{\lambda \in \Sigma_\theta} \left\| \lambda(\lambda - A)^{-1} \right\|_{\mathcal{L}(E_0)} \leqslant c \tag{2.3}$$

thus clarifying the name. If $\theta > \pi/2$, it is well know that an *analytic semigroup* T_A can be associated to any given pseudo-sectorial operator A through the formula

$$T_A(t) = e^{tA} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda - A)^{-1}\, d\lambda, \quad t > 0, \tag{2.4}$$

where the path Γ is given by

$$\Gamma = \left\{ \lambda \in \mathbb{C} \;\middle|\; \arg(\lambda) = \eta,\ |\lambda| \geqslant r \right\} \cup \left\{ \lambda \in \mathbb{C} \;\middle|\; \left|\arg(\lambda)\right| \leqslant \eta, |\lambda| = r \right\}$$

and is oriented counterclockwise. The parameters r and η are chosen such that $r > 0$ and $\eta \in (\pi/2, \theta)$. If the operator A is densely defined, then the semigroup T_A is *strongly continuous*, that is, it satisfies

$$\lim_{t \to 0+} T_A(t)x = x, \quad x \in E_0.$$

Otherwise it is strongly continuous on $\overline{\mathrm{dom}(A)}$.

The most important properties of analytic semigroups (and an equivalent characterization) are

$$\text{(i)} \quad e^{tA}(E_0) \subset \mathrm{dom}(A), \quad t > 0, \tag{2.5}$$

$$\text{(ii)} \quad \left\| tAe^{tA} \right\|_{\mathcal{L}(E_0)} \leqslant c, \quad t \in [0,T],\ T > 0. \tag{2.6}$$

The semigroup T_A is called exponentially decaying if it satisfies

$$\|T_A(t)\|_{\mathcal{L}(E_0)} \leqslant ce^{-\omega t}, \quad t > 0, \tag{2.7}$$

for some $\omega > 0$. The collection of generators A of analytic semigroups satisfying (2.7) for some c is denoted by $\mathcal{H}^-(E_0, \omega)$. Semigroups are useful in the analysis of *abstract Cauchy problems* (ACP)

$$\dot{u} - Au = f(t), \quad u(0) = x. \tag{2.8}$$

If $f \in \mathrm{L}_1(0, T; E_0)$ and $x \in E_0$, a function $u \in \mathrm{C}([0, T], E_0)$ satisfying

$$u(t) = T_A(t)x + \int_0^t T_A(t - \tau) f(\tau)\, d\tau, \quad t \in [0, T], \tag{2.9}$$

is called *mild solution* of (2.8). If the abstract Cauchy problem is nonautonomous, that is, if A depends on the time variable, then mild solutions of

$$\dot{u} - A(t)u = f(t), \quad u(0) = x, \tag{2.10}$$

are given by

$$u(t) = U_A(t, 0)x + \int_0^t U_A(t, \tau) f(\tau)\, d\tau, \tag{2.11}$$

if it can be shown that an *evolution operator* U_A associated to the family A exists. The latter is a two-parameter family

$$\bigl\{U_A(t, \tau) \bigm| t \in [0, T],\ \tau \in [0, t]\bigr\} \subset \mathcal{L}(E_0) \tag{2.12}$$

satisfying

(i) $U_A(t, t) = \mathrm{id}_{E_0}$, $\quad U_A(t, \tau)U_A(\tau, s) = U_A(t, s), \quad 0 \leqslant s \leqslant \tau \leqslant t \leqslant T$,

(ii) $\partial_t U_A(t, \tau)x = A(t)U_A(t, \tau)x, \quad \tau < t \leqslant T,\ x \in E_0$.

Classical results show that such an evolution operator exists on some regularity assumptions on the family A, usually of Hölder type. In [2,23,24], in the case of densely defined family A, it is obtained as the solution to the weakly singular Volterra integral equation

$$U_A(t, s) = e^{(t-s)A(s)} - \int_s^t U_A(t, \tau)\bigl[A(\tau) - A(s)\bigr] e^{(\tau - s)A(s)}\, d\tau. \tag{2.13}$$

Another construction due to Da Prato and Lunardi [5,6,20,22] is based on maximal regularity results for the autonomous abstract Cauchy problem combined with perturbation arguments but does not rely on the family A having dense domains of definition. They obtain

$$U_A(t,s)x = W_A(t,s)x + e^{(t-s)A(s)}x$$

from the solution $w = W_A(\cdot,s)x$ of

$$\dot{w}(t) = A(s)w(t) + \big[A(t) - A(s)\big]x, \quad w(s) = x. \tag{2.14}$$

Optimal or maximal regularity results for (2.8) or (2.10) can be described as follows: find spaces S and E_0 such that (2.8) or (2.10) possess a unique solution $u \in S((0,T], E_0)$ with

$$\dot{u}, Au \in S\big((0,T], E_0\big) \quad \text{and} \quad \|\dot{u}\|_{S((0,T],E_0)} + \|Au\|_{S((0,T],E_0)} \leqslant c\|f\|_{S((0,T],E_0)}. \tag{2.15}$$

The name obviously refers to the fact that u enjoys as much regularity as is conceivably possible by virtue of it satisfying (2.8) or (2.10), respectively. It is known that restrictions apply to the choice of S and E_0 for maximal regularity to hold. Counter examples are given in [20,21].

Fractional powers and interpolation spaces play an important role in the theory of abstract Cauchy problems of parabolic type. Given a pseudo-sectorial operator A it is always possible to define its *fractional powers* $(-A)^\rho$, $\rho > 0$, as the inverses of the bounded operators

$$(-A)^{-\rho} := \frac{1}{\Gamma(\rho)} \int_0^\infty t^{\rho-1} e^{tA}\,dt \tag{2.16}$$

defined on their range, that is, with

$$\mathrm{dom}\big((-A)^\rho\big) = R\big((-A)^{-\rho}\big).$$

For $\rho = 1$, (2.16) simply gives the resolvent of A as the Laplace transform of the semigroup e^{tA}. The *interpolation spaces*

$$E_\alpha^p = D_A(\alpha,p), \quad \alpha \in (0,1),\ p \in [1,\infty] \text{ or } (\alpha,p) = (1,\infty),$$

are defined by

$$E_\alpha^p := \big\{x \in E_0 \,\big|\, \big[t \mapsto v(t) := \big\|t^{1-\alpha-1/p} A e^{tA} x\big\|\big] \in \mathrm{L}_p(0,1)\big\}, \tag{2.17}$$

$$\|x\|_{E_\alpha^p} = \|x\| + [x]_{\alpha,p} := \|x\| + \|v\|_{\mathrm{L}_p(0,1)}, \tag{2.18}$$

with the standard convention that $1/\infty = 0$. The domain $\mathrm{dom}((-A)^\alpha)$ endowed with the graph norm $\|x\|_{D_\alpha} = \|(-A)^\alpha x\|$ is denoted by D_α or $D((-A)^\alpha)$. It can be conveniently sandwiched between interpolation spaces

$$E_\alpha^1 \hookrightarrow D_\alpha \hookrightarrow E_\alpha^\infty, \quad \alpha \in (0,1), \tag{2.19}$$

and satisfies the interpolation property

$$\left\|(-A)^{\alpha}x\right\| \leqslant c\|Ax\|^{\alpha}\|x\|^{1-\alpha}.$$

One of the main reasons to consider maximal regularity results is dealing with fully nonlinear Cauchy problems like

$$\dot{u} = f(t,u), \quad t > 0,\ u(0) = u_0 \in E_0, \tag{2.20}$$

in E_0 or quasilinear problems like

$$\dot{u} - A(t,u)u = f(t,u), \quad t > 0,\ u(0) = u_0 \in E_0. \tag{2.21}$$

The basic idea is to linearize (2.20) (or (2.21)) in u_0 and localize about $t = 0$ to obtain

$$\dot{u} - A(0)u = G(u), \quad t > 0, \tag{2.22}$$

where $A(0) = DF(0,u_0)$ and $DG(0,u_0) = 0$ and solve it by a fixed-point argument in a small time interval. Repeating the procedure the solution can be extended to its maximal interval of existence. This can only be done if maximal regularity results hold which are essentially equivalent to $\partial_t - A(0)$ being invertible. In spite of the simplicity of this idea, its implementation is rather involved and needs the introduction of spaces of singular Hölder continuous functions in order to successfully deal with the kernel singularity (2.6) in the origin. A comprehensive exposition of this theory is given in [20], where the author also presents a variety of examples which clearly attest to its wide range of applicability. One may also consult [4].

The focus of this paper is on *singular* abstract Cauchy problems like

$$\dot{u} - A(t)u = f(t), \quad t > 0, \tag{2.23}$$

in a Banach space E_0. The family A is allowed to behave singularly in the origin. In this case (2.23) is the appropriate model problem since there is no autonomous counterpart to speak of. One of the main goals of the paper is to find abstract but flexible conditions on the singular family A which allow for the construction of an associated evolution operator and eventually lead to maximal regularity results for (2.23). Again the prime applications would be fully nonlinear counterparts of (2.23), which can be written as

$$\dot{u} - A(t,u)u = f(t,u), \quad t > 0. \tag{2.24}$$

In this case the problem needs not only to be linearized as in the regular case but also to be expanded in the singularity in order to capture its leading order behavior and perturb around it. A notation closer to (2.21) rather than (2.20) is used to draw reader's attention to the singular behavior in the origin of the generator families $A(\cdot,u)$ involved. A class of free boundary problems, considered in [18] for the first time in higher dimensions and by many authors in a one-dimensional setting [1,7,8,10–13,16], leads naturally to equations of singular type and motivates the study presented here.

Singular families of type

$$A(t) = \frac{A}{t^k}, \quad k > 1, \tag{2.25}$$

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and natural generalizations thereof were considered in [17] where an associated evolution operator was constructed by using (2.13) and results for a corresponding class of quasilinear equations were obtained. In the purely linear case, they were also considered by [14,25] were operator sum/product techniques were used in a L_p-setting.

Here the aim is to extend those results to obtain maximal regularity in spaces of (singular) Hölder continuous functions for a nondensely defined family of generators and for a generalization of (2.25) in which the singularity is allowed to affect the operator in an anisotropic way. A simple example is given by

$$A(t) = B + \frac{C}{t^k} \tag{2.26}$$

where B, C are given generators of analytic semigroups in the sense explained of (2.4). In applications the operators B and C are usually differential operators acting on distinct spacial variables. However, they typically do not commute as they in general have non-constant coefficients. All the results mentioned above do not apply to the setting of this paper and only those in [17] could be adapted to it but merely in the event that B and C were commuting operators. This latter case is not of much interest since it never occurs in practical applications.

Moreover the conditions previously used in [17] are ad-hoc in that they rely on the singularity being of a given explicit type (power type) in order to construct the associated evolution operator. In [14,25] no evolution operator is constructed but the singularity is assumed to be given by a simple function of time.

Here a more abstract condition is obtained which seems quite natural in the construction of the evolution operator as the proofs will show and which allows for a wider class of singularities including all those previously considered in the mentioned papers. These conditions read

$$\text{(i)} \quad A(t) \in \mathcal{H}^-(E_0, \omega), \quad t > 0, \tag{2.27}$$

$$\text{(ii)} \quad \big\| \big[A(t) - A(s)\big] A^{-1}(\tau) \big\|_{\mathcal{L}(E_0)} \leqslant c \frac{t-s}{t} \quad \text{and}$$
$$\big\| \big[A(t) - A(s)\big] (-A)^{-\rho}(\tau) \big\|_{\mathcal{L}(E_0)} \leqslant c(t-s), \tag{2.28}$$

$$\text{(iii)} \quad \lim_{t \to 0} A^{-1}(t) = 0, \tag{2.29}$$

for some $\rho \in (1, 2)$ and $0 < \tau \leqslant s \leqslant t \leqslant T$. As an example one can consider $A(t) = B + \frac{C}{t^2}$ satisfying (2.27) and such that C is an invertible pseudo-sectorial operator, then (2.28)–(2.29) are easily seen to be satisfied.

For the sake of completeness we observe that maximal regularity results have been obtained, in the regular case not considered here, in a variety of other function spaces other than singular Hölder spaces and with the base space E_0 substituted in particular by interpolation spaces of type (2.17). The interested reader may consult [2,9,19,20,22] for instance. Some of these results cannot be obtained for the singular (anisotropic case), whereas others do not fit the requirements imposed by the applications to free boundary problems. In spite of the fact that they would be of theoretical and possibly of practical interest, they are not considered in this paper.

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3. Construction of the evolution operator

In order to construct the evolution operator U associated to a singular family satisfying (2.27)–(2.29) it is necessary to work in spaces of singular Hölder continuous functions.

Definition 3.1. Let $\alpha \in (0,1)$ and $T > 0$ be given and let E be a Banach space

$$\mathrm{C}^\alpha_\alpha\big((0,T],E\big) := \big\{v \in \mathrm{B}\big((0,T],E\big) \bigm| \big[t \mapsto t^\alpha v(t)\big] \in \mathrm{C}^\alpha\big((0,T],E\big)\big\}. \tag{3.1}$$

Endowed with the norm given by

$$\|v\|_{\alpha,s} := \|v\|_\infty + [v]_{\alpha,s}$$

this space becomes a Banach space. Hereby we denoted by $[\cdot]_{\alpha,s}$ the weighted Hölder seminorm

$$[v]_{\alpha,s} := \big[(\cdot)^\alpha v\big]_\alpha := \sup_{0<t\neq s\leqslant T} \frac{\|t^\alpha v(t) - s^\alpha v(s)\|_E}{|t-s|^\alpha}$$

where $[\cdot]_\alpha$ denotes the regular Hölder seminorm. We shall also make use of the space

$$\mathrm{C}^\alpha_0\big((0,T],E\big) := \big\{v \in \mathrm{C}^\alpha\big([0,T],E\big) \bigm| v(0) = 0\big\}. \tag{3.2}$$

For $\beta \in (0,1)$ the additional space

$$\begin{aligned}\mathrm{C}^\alpha_{\alpha,\beta}\big((0,T],E\big) := \big\{v : (0,T] \to E \bigm| &\big[t \mapsto t^\beta v(t)\big] \in \mathrm{B}\big((0,T],E\big),\\ &\big[t \mapsto t^{\alpha+\beta} v(t)\big] \in \mathrm{C}^\alpha\big((0,T],E\big)\big\}\end{aligned} \tag{3.3}$$

is also defined and endowed with its natural norm

$$\|v\|_{\alpha,\beta} = \big\|(\cdot)^\beta v\big\|_\infty + \big[(\cdot)^{\alpha+\beta} v\big]_\alpha.$$

The space

$$\mathrm{B}_\beta\big((0,T],E\big) := \big\{v : (0,T] \to E \bigm| \big[t \mapsto t^\beta v(t)\big] \in \mathrm{B}\big((0,T],E\big)\big\}$$

will also be useful.

Fix $x \in E_0$. Then it is natural to look for $U(\cdot,s)x$ as the solution of

$$\dot u = A(t)u, \quad t \in (s,T],\ u(s) = x, \tag{3.4}$$

which is rewritten as

$$\dot w = A(t)w + \big[A(t) - A(s)\big]e^{(t-s)A(s)}x, \quad t \in (s,T],\ u(s) = 0, \tag{3.5}$$

by setting $w(t) := u(t) - e^{(t-s)A(s)}x$. If (3.5) can be solved and denoting by $W(\cdot, s)x$ its solution in that case, the evolution operator is then simply given by

$$U(t,s) = W(t,s) + e^{(t-s)A(s)}. \tag{3.6}$$

The next theorem establishes existence for (3.5) in the space (3.3).

Theorem 3.2. *Assume that A satisfies* (2.27)–(2.29) *with $\rho \in (1,2)$ and let $f \in \mathrm{C}^{\alpha}_{\alpha,\rho-1}((s,T],E)$ for $\alpha \in (0,1)$, $s \in (0,T)$. Then the solution w of*

$$\dot{w} = A(t)w + \underbrace{\big[A(t) - A(s)\big]e^{(t-s)A(s)}x}_{=:g_s(t)} + f(t), \quad t \in (s,T],\ w(0) = 0,$$

satisfies

$$w, \dot{w}, A(s)w \in \mathrm{C}^{\alpha}_{\alpha,\rho-1}\big((s,T],E_0\big), \qquad \dot{w} \in \mathrm{B}_{\alpha+\rho-1}\big((s,T],E^{\infty}_{\alpha}\big), \tag{3.7}$$

$$\|Aw\|_{\mathrm{C}^{\alpha}_{\alpha,\rho-1}E_0} + \|\dot{w}\|_{\mathrm{C}^{\alpha}_{\alpha,\rho-1}E_0} + \|\dot{w}\|_{\mathrm{B}_{\alpha+\rho-1}E^{\infty}_{\alpha}} \leqslant c\big(\|x\| + \|f\|_{\mathrm{C}^{\alpha}_{\alpha,\rho-1}E_0}\big). \tag{3.8}$$

Proof. The solution is constructed as the unique fixed-point of Φ in the space $\mathrm{C}^{\alpha}_{\alpha,\rho-1}((s,T], D(A(s)))$ where $\Phi(v)$ is defined as the solution of

$$\dot{w} = A(s)w + \big[A(t) - A(s)\big]\big[v + e^{(t-s)A(s)}x\big] + f(t), \quad t \in (s,T],\ w(0) = 0.$$

Step 1. First it is checked that $g_s \in \mathrm{C}^{\alpha}_{\alpha,\rho-1}((s,T],E_0)$. In fact

$$\begin{aligned}\big\|\big[A(t) - A(s)\big]e^{(t-s)A(s)}x\big\| &= \big\|\big[A(t) - A(s)\big]\big(-A(s)\big)^{-\rho}\big(-A(s)\big)^{\rho}e^{(t-s)A(s)}x\big\| \\ &\leqslant c\frac{1}{(t-s)^{\rho-1}},\end{aligned}$$

which gives $g_s \in \mathrm{B}_{\rho-1}((s,T],E_0)$. Next take $0 < s < s+\varepsilon \leqslant r \leqslant t \leqslant T$ and consider

$$\begin{aligned}&\varepsilon^{\alpha+\rho-1}\big\|g_s(t) - g_s(r)\big\| \\ &\quad\leqslant \varepsilon^{\alpha+\rho-1}\big\|\big[A(t) - A(r)\big]\big(-A(s)\big)^{-\rho}\big(-A(s)\big)^{\rho}e^{(t-s)A(s)}x\big\| \\ &\qquad + \varepsilon^{\alpha+\rho-1}\left\|\big[A(r) - A(s)\big]\int_{r-s}^{t-s} A(s)e^{\sigma A(s)}x\,d\sigma\right\| \\ &\quad\leqslant c\varepsilon^{\alpha+\rho-1}(t-r)\frac{1}{(t-s)^{\rho}}\|x\| + c\varepsilon^{\alpha+\rho-1}(r-s)\int_{r-s}^{t-s}\frac{1}{\sigma^{1+\rho}}\,d\sigma \\ &\quad\leqslant c\|x\|\left[(t-r)^{\alpha}(t-r)^{1-\alpha}\frac{\varepsilon^{\alpha+\rho-1}}{t-s} + \varepsilon^{\alpha+\rho-1}(r-s)\frac{(t-s)^{\rho} - (r-s)^{\rho}}{(r-s)^{\rho}(t-s)^{\rho}}\right] \\ &\quad\leqslant c\|x\|(t-r)^{\alpha},\end{aligned}$$

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since $(t-s)^\rho-(r-s)^\rho \leqslant c(t-s)^{\rho-1}(t-r)$. It is therefore seen that

$$g_s \in \mathrm{C}^\alpha_{\alpha+\rho-1}\big((s,T],E_0\big) \quad \text{and} \quad \|g_s\|_{\alpha,\rho-1} \leqslant c\|x\|,$$

for a constant c which does not depend on s.

Step 2. Next it is shown that Φ is a contractive self-map on $\mathrm{C}^\alpha_{\alpha,\rho-1}((s,s+\delta],D(A(s)))$ provided $\delta>0$ is small enough. In order to do so, it is enough to show contractivity together with

$$\big[A(\cdot)-A(s)\big]v \in \mathrm{C}^\alpha_{\alpha+\rho-1}\big((s,T],E_0\big) \tag{3.9}$$

because the existence and regularity of the solution $\Phi(v)$ then follows from known maximal regularity results for the regular case (cf. [20, Theorem 4.3.7]). In order to show (3.9) observe first that

$$\begin{aligned}
&\big\|(t-s)^{\rho-1}\big[A(t)-A(s)\big]\big(-A(s)\big)^{-1}A(s)v(t)\big\| \\
&\quad\leqslant \frac{t-s}{t}\big\|(t-s)^{\rho-1}A(s)v(t)\big\| \leqslant \frac{\delta}{s}\|v\|_{\mathrm{B}_{\rho-1}D(A(s))}.
\end{aligned}$$

Next take $0<s<s+\varepsilon\leqslant r\leqslant t\leqslant T$ and consider

$$\begin{aligned}
&\varepsilon^{\alpha+\rho-1}\big\|\big[A(t)-A(s)\big]v(t)-\big[A(r)-A(s)\big]v(r)\big\| \\
&\quad\leqslant \varepsilon^{\alpha+\rho-1}\big\|\big[A(t)-A(r)\big]v(t)\big\|+\varepsilon^{\alpha+\rho-1}\big\|\big[A(r)-A(s)\big]\big(v(t)-v(r)\big)\big\| \\
&\quad\leqslant \varepsilon^{\alpha+\rho-1}\frac{t-r}{t}\frac{1}{\varepsilon^{\rho-1}}\|v\|_{\mathrm{B}_{\rho-1}D(A(s))}+\frac{r-s}{r}(t-r)^\alpha\|v\|_{\mathrm{C}^\alpha_{\alpha+\rho-1}D(A(s))} \\
&\quad\leqslant c\frac{\delta}{s}\|v\|_{\mathrm{C}^\alpha_{\alpha+\rho-1}D(A(s))},
\end{aligned}$$

for a constant c independent of s. This gives (3.9) and shows that

$$\big\|\big[A(\cdot)-A(s)\big]v\big\|_{\alpha,\rho-1,E_0} \leqslant c\frac{\delta}{s}\|v\|_{\alpha,\rho-1,D(A(s))}$$

on the interval $(s,s+\delta]$. Latter estimate also gives that

$$\big\|\Phi(v_1)-\Phi(v_2)\big\|_{\alpha,\rho-1,D(A(s))} \leqslant c\frac{\delta}{s}\|v_1-v_2\|_{\alpha,\rho-1,D(A(s))}$$

in view of [20, Theorem 4.3.7] and the linearity of the equation. It is then clear that Φ is a contraction for $\delta \ll 1$ and it is easy to obtain the inequality

$$\|w\|_{\alpha,\rho-1,D(A(s))} \leqslant c\|g_s\|_{\alpha,\rho-1,E_0} \leqslant c\|x\|$$

for the unique solution w for a constant c independent of s. By using further results from the regular theory [20, Proposition 6.1.3] the solution can be continued to the full interval maintaining the inequality. □

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It follows that W (for $f \equiv 0$) satisfies the slightly better inequality

$$\left\| A(\tau)W(t,\tau)\right\|_{\mathcal{L}(E_0)} \leqslant c\frac{1}{(t-\tau)^{\rho-1}}, \quad t \in (s,T]. \tag{3.10}$$

Taking (3.6) into account it is concluded that

$$\left\| A(\tau)U(t,\tau)\right\|_{\mathcal{L}(E_0)} \leqslant c\frac{1}{(t-\tau)}, \quad t \in (s,T]. \tag{3.11}$$

It is important to point out that the constant appearing in both estimates is independent of τ as follows from the proof of Theorem 3.2.

Remark 3.1. In the regular case there is no restriction in the choice of the exponent $\rho - 1$, whereas here it is determined by the singularity through (2.28).

Corollary 3.3. *Let A satisfy* (2.27)–(2.29). *Then there is a unique evolution operator U associated to A defined for $T \geqslant t \geqslant \tau > 0$. It can be extended to $\tau = 0$ by setting*

$$U(t,0) = 0, \quad 0 < t \leqslant T.$$

Proof. The claim follows from

$$\begin{aligned}\left\|U(t,\tau)\right\|_{\mathcal{L}(E_0)} &\leqslant \left\|A^{-1}(\tau)\right\|_{\mathcal{L}(E_0)}\left\|A(\tau)U(t,\tau)\right\|_{\mathcal{L}(E_0)} \\ &\leqslant c\frac{1}{t-\tau}\left\|A^{-1}(\tau)\right\|_{\mathcal{L}(E_0)} \to 0 \quad (\tau \to 0)\end{aligned}$$

in view of (2.29). □

The evolution operator allows to characterize solutions of (2.23) through the classical variation-of-constant-formula.

Proposition 3.4. *Lt $f \in \mathrm{L}_1((0,T),E_0)$. Then any bounded mild solution u of* (2.23) *is given by*

$$u(t) = \int_0^t U(t,\tau)f(\tau)\,d\tau.$$

Proof. The assumption on the function f together with ensures the existence of the variation of constant integral. In the event that the solution u to (2.23) exists and is bounded, it coincides with the solution u_δ of (2.23) on $t \geqslant \delta$ corresponding to the initial condition $u_\delta(\delta) = u(\delta)$. It is then necessarily given by

$$u(t) = u_\delta(t) = U(t,\delta)u(\delta) + \int_\delta^t U(t,\tau)f(\tau)\,d\tau, \quad t \geqslant \delta.$$

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Since the convolution integral exists for $\delta = 0$ the second term converges to (3.4). By assumption the first can be estimated as follows

$$\left\| U(t,\delta)u(\delta) \right\|_{E_0} \to 0 \quad (\delta \to 0,\ t > 0)$$

by virtue of the solution's boundedness and Corollary 3.3. The function u is therefore a mild solution of (2.23) for $t > 0$. $\square$

Remark 3.2.

(a) It now becomes clear why (2.23) is formulated without any initial condition. Bounded solutions naturally emanate from 0.
(b) If additional information is available about the rate of vanishing of $A^{-1}(t)$ in $t = 0$, it is possible to weaken the assumption on f to

$$\left[t \to t^p f(t)\right] \in \mathrm{L}_1\big((0,T), E_0\big)$$

for an appropriate power $p > 1$.

4. Maximal regularity

In order to prove maximal regularity results for (2.23) a couple of lemmata are needed.

Lemma 4.1. *Let $x \in E_0$ and A be the generator of a not necessarily strongly continuous analytic semigroup. Then $\int_0^t e^{sA} x\, ds \in D(A)$ and*

$$A \int_0^t e^{sA} x = e^{tA} x - x.$$

Proof. The proof would be completely obvious if the semigroup were strongly continuous. On the given assumptions it needs a little more care but a proof can be found in [20]. $\square$

Lemma 4.2. *Assume that A satisfies assumptions* (2.27)–(2.29). *Then*

$$\left\| A(t)\left[e^{(t-\tau)A(\tau)} - e^{(t-\tau)A(t)}\right] \right\|_{\mathcal{L}(E_0)} \leqslant c\frac{1}{t}, \quad 0 < \tau < t \leqslant T.$$

Proof. Since

$$e^{(t-\tau)A(\tau)} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda(t-\tau)} \big(\lambda - A(\tau)\big)^{-1} d\lambda$$

the estimand can be rewritten as

$$-\frac{1}{2\pi i} \int_\Gamma e^{\lambda(t-\tau)} A(t)\big(\lambda - A(t)\big)^{-1} \left[A(\tau) - A(t)\right] A^{-1}(\tau) A(\tau) \big(\lambda - A(\tau)\big)^{-1} d\lambda.$$

Assumptions (2.27) and (2.28) give

$$\left\|A(t)\big(\lambda - A(t)\big)^{-1}\big[A(\tau) - A(t)\big]A^{-1}(\tau)A(\tau)\big(\lambda - A(\tau)\big)^{-1}\right\|_{\mathcal{L}(E_0)} \leqslant c\frac{t-\tau}{t}$$

and the claim follows by direct estimation of the integral. □

Theorem 4.3. *Assume that A satisfies assumptions* (2.27)–(2.29) *and let $f \in \mathrm{C}_0^\alpha((0,T], E_0)$ for some $\alpha \in (0,1)$. Then the solution u of* (2.23) *given by* (3.4) *on* $(0,T]$ *satisfies*

$$\dot u,\, Au \in \mathrm{C}_0^\alpha \quad \textit{and} \quad \|\dot u\|_\alpha + \|Au\|_\alpha \leqslant c\|f\|_\alpha.$$

Proof. *Step 1.* First Hölder continuity in the origin is shown. To that end it is enough to show that

$$\left\|A(t)\int_0^t U(t,\tau)f(\tau)\,d\tau\right\| \leqslant ct^\alpha$$

since then the equation gives the corresponding estimate for $\dot u(t)$. Using the decomposition (3.6) of the evolution operator it is seen that

$$\begin{aligned}\int_0^t U(t,\tau)f(\tau)\,d\tau &= \int_0^t W(t,\tau)f(\tau)\,d\tau + \int_0^t e^{(t-\tau)A(\tau)}\big[f(\tau) - f(t)\big]\,d\tau \\ &\quad + \int_0^t \big[e^{(t-\tau)A(\tau)} - e^{(t-\tau)A(t)}\big]f(t)\,d\tau + \int_0^t e^{(t-\tau)A(t)}f(t)\,d\tau \\ &= I + II + II + IV.\end{aligned}$$

The various terms can then be estimated as follows

$$\big\|A(t)I\big\| \leqslant c\int_0^t \frac{\tau^\alpha}{t-\tau}\,d\tau \leqslant ct^\alpha \quad \text{by (3.10)},$$

$$\big\|A(t)II\big\| \leqslant c\int_0^t \frac{1}{(t-\tau)^{1-\alpha}}\,d\tau = ct^\alpha \quad \text{by assumption},$$

$$\big\|A(t)III\big\| \leqslant c\int_0^t \frac{1}{t^{1-\alpha}}\,d\tau = ct^\alpha \quad \text{by Lemma 4.2.}$$

As for the last one has $A(t)IV = e^{tA(t)}f(t) - f(t)$ by Lemma 4.1 which gives the desired estimate by the assumptions on f.

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Step 2. Away from the origin, solution properties should not deviate from the regular case. This is in fact confirmed by the following argument. It follows from the previous step that

$$u(t)=\int_0^t U(t,\tau)\,d\tau\in D\big(A(t)\big),\quad t\in(0,T].$$

Thus $u(\delta)\in D(A(\delta))$ and, using Theorem 3.2 one gets that

$$\dot u(\delta)=A(\delta)u(\delta)+f(\delta)\in E_\infty^\alpha$$

and that

$$\big\|u(\delta)\big\|_{D(A(\delta))}\leqslant c\|f\|_{\alpha,\rho-1}\leqslant c\|f\|_\alpha\quad\text{and}$$
$$\big\|A(\delta)u(\delta)+f(\delta)\big\|_{E_\infty^\alpha}\leqslant c\|f\|_{\alpha,\rho-1}\leqslant c\|f\|_\alpha.$$

The embedding inequalities are a consequence of

$$\big\|t^{\rho-1}f(t)\big\|\leqslant ct^{\rho-1+\alpha}[f]_\alpha\quad\text{and of}$$
$$\begin{aligned}\left\|\frac{t^{\alpha+\rho-1}f(t)-s^{\alpha+\rho-1}f(s)}{(t-s)^\alpha}\right\|&\leqslant\left\|\frac{t^{\alpha+\rho-1}-s^{\alpha+\rho-1}}{(t-s)^\alpha}f(t)\right\|+\left\|s^{\alpha+\rho-1}\frac{f(t)-f(s)}{(t-s)^\alpha}\right\|\\&\leqslant(t-s)^{\min(\alpha+\rho-1,1)}t^\alpha[f]_\alpha+s^{\alpha+\rho-1}[f]_\alpha.\end{aligned}$$

Thus, using [20, Theorem 4.3.1(iii)], it follows that $\dot u$, $Au\in \mathrm{C}^\alpha([\delta,T],E_0)$ for any given $\delta>0$ and

$$\|\dot u\|_\alpha+\|Au\|_\alpha\leqslant c\|f\|_\alpha$$

which gives Hölder continuity everywhere away from the origin. □

Theorem 4.4. *Assume that A satisfies assumptions* (2.27)–(2.29) *and let $f\in \mathrm{C}_\alpha^\alpha((0,T],E_0)$ for some $\alpha\in(0,1)$. Then the solution u of* (2.23) *given by* (3.4) *on* $(0,T]$ *satisfies*

$$\dot u,\ Au\in \mathrm{C}_\alpha^\alpha\quad\text{and}\quad\|\dot u\|_{\alpha,s}+\|Au\|_{\alpha,s}\leqslant c\|f\|_{\alpha,s}.$$

Proof. *Step 1*. First consider regularity in the origin. From (3.6) it follows that

$$A(t)\int_0^t U(t,\tau)f(\tau)\,d\tau=A(t)\int_0^t W(t,\tau)f(\tau)\,d\tau+\int_0^t e^{(t-\tau)A(\tau)}f(\tau)\,d\tau=I_1+I_2.$$

The first term leads to

$$\|I_1\|\leqslant\int_0^t\frac{1}{(t-\tau)^{\rho-1}}\,d\tau=ct^{2-\rho}.$$

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I_2 needs to be further split

$$\begin{aligned}I_2 &= A(t)\int_0^t U(t,\tau)\big[f(\tau)-f(t)\big]\,d\tau + A(t)\int_0^t \big[e^{(t-\tau)A(\tau)}-e^{(t-\tau)A(t)}\big]f(t)\,d\tau\\ &\quad + e^{tA(t)}f(t)-f(t)\\ &= I+II+III,\end{aligned}$$

where Lemma 4.1 was used. The estimate for III follows. As for II one has

$$\|II\| \leqslant c\int_0^t \frac{t-\tau}{t}\frac{1}{t-\tau}\,d\tau \leqslant c$$

by Lemma 4.2. Finally III gives

$$\|III\| \leqslant c\int_0^t \frac{1}{(t-\tau)^{1-\alpha}}\frac{1}{\tau^\alpha}\,d\tau = c\int_0^1 \frac{1}{(1-\sigma)^{1-\alpha}}\frac{1}{\sigma^\alpha}.$$

Step 2. Away from the origin it is again possible to argue as in the regular case. Using

$$\begin{gathered}u(\delta)\in D\big(A(\delta)\big), \qquad A(\delta)u(\delta)+f(\delta)\in E_\infty^\alpha \quad \text{and}\\ \big\|u(\delta)\big\|_{D(A(\delta))}+\big\|A(\delta)u(\delta)+f(\delta)\big\|_{E_\infty^\alpha}\leqslant c\|f\|_{\alpha,\rho-1}.\end{gathered}$$

It follows again from [20, Theorem 4.3.1(iii)] that

$$\dot u, Au \in \mathrm{C}^\alpha\big([\delta,T],E_0\big)$$

and that

$$\begin{aligned}\|\dot u\|_{\alpha,[\delta,T]}+\|Au\|_{\alpha,[\delta,T]} &\leqslant c\big(\|f\|_{\alpha,[\delta,T]}+\big\|u(\delta)\big\|_{D(A(\delta))}+\big\|A(\delta)u(\delta)+f(\delta)\big\|_{E_\infty^\alpha}\big)\\ &\leqslant c\frac{1}{\delta^\alpha}\|f\|_{\alpha,s}\end{aligned}$$

because of the embedding

$$\mathrm{C}^\alpha_{\alpha,\rho-1}\big((0,T],E_0\big)\hookrightarrow \mathrm{C}^\alpha_\alpha\big((0,T],E_0\big)$$

and since

$$\|f\|_{\alpha,[\delta,T]}\leqslant c\frac{1}{\delta^\alpha}\|f\|_{\alpha,s}$$

as can be easily checked. The desired estimate is therefore obtained. □

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Remark 4.1. It should be pointed out that, whereas conditions (2.27)–(2.29) are quite general, they, however, exclude singular families like

$$A(t) = \frac{A}{t^\beta}$$

with $\beta \leqslant 1$ and the generator A of an analytic exponentially decaying semigroup T_A. This is not due to (2.27)–(2.29) being too restrictive but rather to the fact that the regularity results obtained here are not valid in that case. This follows from the fact that the inequality

$$\left\| A(\tau)U(t,\tau)\right\|_{\mathcal{L}(E_0)} \leqslant \left\| \frac{A}{\tau^\beta} T_A\left(\int_\tau^t \sigma^{-\beta}\, d\sigma\right)\right\|_{\mathcal{L}(E_0)} \leqslant c\frac{t^\beta}{\tau^\beta}\frac{1}{t-\tau}$$

does not yield the needed

$$\left\| A(\tau)U(t,\tau)\right\|_{\mathcal{L}(E_0)} \leqslant c\frac{1}{t-\tau}$$

with a constant independent of τ. If follows that this case has to be treated differently. This example also shows that condition (2.28) cannot be weakened to an analogous singular Hölder condition.

5. An example

In this last section an example is considered of a initial boundary value problem on a moving domain which undergoes an initial dimensional change. It is the latter that will eventually lead to a singular evolution equation of type (2.23). Let a function $0 < \inf(\varphi) \leqslant \varphi \in \mathrm{BUC}^{2+\beta}(\mathbb{R}^{n-1}, \mathbb{R})$ be given. Consider the diffusion equation

$$\dot{u} - \Delta u = 0, \qquad (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} \quad \text{with } 0 < y < t\varphi(x),\ t > 0, \tag{5.1}$$

complemented by the boundary conditions

$$u(t, x, 0) = g(x), \quad (x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \tag{5.2}$$

$$\partial_\nu u\big(t, x, t\varphi(x)\big) = h(x), \quad (x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \tag{5.3}$$

for some given $g \in \mathrm{BUC}^{2+\beta}(\mathbb{R}^{n-1}, \mathbb{R})$ and $h \in \mathrm{BUC}^{1+\beta}(\mathbb{R}^{n-1}, \mathbb{R})$. Problem (5.1)–(5.3) is a parabolic initial boundary value problem in the space–time wedge

$$\bigcup_{t \leqslant 0} \{t\} \times [0 < y < t\varphi].$$

Remark 5.1. For the free boundary problems mentioned earlier in the paper the upper boundary of the domain would be given by a function $\varphi(t, x)$ which is itself an unknown of the problem and satisfies an additional evolution equation with initial condition $\varphi(0, \cdot) \equiv 0$, thus introducing a singularity into the problem via the change of variable (5.4).

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In order to apply the abstract results derived in the previous sections, the problem needs to be reformulated in a new set of variables

$$(\tau, \xi, \eta) := \left(t, x, \frac{y}{t\varphi(x)}\right). \tag{5.4}$$

If one rewrites the equations in the new variables using the names of the old variables, one obtains

$$\begin{aligned}
&\dot{u} - \Delta_x u - \frac{1 + t^2|\nabla\varphi|^2}{t^2\varphi^2}\partial_{yy}u = \frac{y}{t}\partial_y u + 2\left(\frac{\nabla\varphi}{\varphi}\middle|\partial_y \nabla u\right) + \frac{\varphi\Delta\varphi - |\nabla\varphi|^2}{\varphi^2}\partial_y u,\\
&\qquad (x, y) \in S,\ t > 0,\\
&u(t, x, 0) = g(x), \quad x \in \mathbb{R}^{n-1},\ t > 0,\\
&\partial_y u(t, x, 1) = \frac{t\varphi}{1 + t|\nabla\varphi|^2}\Big[h\sqrt{1 + |\nabla\varphi|^2} + \big(\nabla\varphi \big| \nabla_x u(t, x, 1)\big)\Big], \quad x \in \mathbb{R}^{n-1},\ t > 0,
\end{aligned}$$

for $S = \mathbb{R}^{n-1} \times (0, 1)$. As the general case is included in the forthcoming analysis [15] of free boundary problems, the simplifying assumption $\varphi \equiv 1$ is now made which leads to the simpler system

$$\dot{u} - \Delta_x u - \frac{1}{t^2}\partial_{yy}u - \frac{y}{t}\partial_y u = 0, \quad (x, y) \in S,\ t > 0, \tag{5.5}$$

$$u(t, x, 0) = g(x), \quad x \in \mathbb{R}^{n-1},\ t > 0, \tag{5.6}$$

$$\partial_y u(t, x, 1) = th(x), \quad x \in \mathbb{R}^{n-1},\ t > 0. \tag{5.7}$$

The solution of (5.5)–(5.7) can be sought in the form

$$u(t, x, y) = v(t, x, y) + \big(\mathcal{R}_D(t)g\big)(x) + t\big(\mathcal{R}_N(t)\big)h(x), \quad (t, x, y) \in (0, \infty) \times S,$$

where the function v satisfies the equation

$$\dot{v} - \Delta_x v - \frac{1}{t^2}\partial_{yy}v - \frac{y}{t}\partial_y v = \left[\frac{y}{t}\partial_y - \partial_t\right]\big(\mathcal{R}_D(t)g + t\mathcal{R}_N(t)h\big) \tag{5.8}$$

complemented with homogeneous boundary conditions, i.e. with

$$v(t, x, 0) = 0 = \partial_y v(t, x, 1). \tag{5.9}$$

The functions $\mathcal{R}_D(t)g$ and $\mathcal{R}_N(t)h$ are defined as follows

$$\mathcal{R}_D(t)g = \mathcal{F}^{-1}\frac{\cosh(t|\xi|(1 - y))}{\cosh(t|\xi|)}\mathcal{F}g, \tag{5.10}$$

$$\mathcal{R}_N(t)h = \mathcal{F}^{-1}\frac{\sinh(t|\xi|y)}{t|\xi|\cosh(t|\xi|)}\mathcal{F}h, \tag{5.11}$$

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and satisfy

$$-\Delta_x u - \frac{1}{t^2}\partial_{yy} u = 0,$$

complemented with the boundary conditions

$$u(t,\cdot,0) = g, \quad u_y(t,\cdot,1) = 0 \quad \text{for } u = R_D(t)g,$$
$$u(t,\cdot,0) = 0, \quad u_y(t,\cdot,1) = th \quad \text{for } u = tR_N(t)h,$$

respectively. Setting $E_0 = \mathrm{BUC}^\beta(\mathbb{R}^{n-1}, \mathrm{C}([0,1]))$ it is possible to use the results collected in the previous sections to obtain well-posedness of (5.8)–(5.9) in the space $\mathrm{C}^\alpha_\alpha((0,T], E_0)$ for $\alpha \in (0,1)$.

Lemma 5.1. *The function* $f := [\frac{y}{t}\partial_y - \partial_t][\mathcal{R}_D(t)g + t\mathcal{R}_N(t)h]$ *satisfies*

$$f \in \mathrm{C}^\alpha_\alpha\big((0,T], E_0\big).$$

Proof. A direct computation shows that

$$\begin{aligned}\mathcal{F}f = {}& \big[(1-y)\sinh\big(t|\xi|(1-y)\big) + \tanh\big(t|\xi|\big)\cosh\big(t|\xi|(1-y)\big)\big]\frac{1}{\cosh(t|\xi|)}|\xi|\mathcal{F}g \\ & - \big[y\cosh\big(t|\xi|y\big) - \tanh\big(t|\xi|\big)\sinh\big(t|\xi|y\big)\big]\frac{1}{\cosh(t|\xi|)}\mathcal{F}h \\ & - y\frac{\sinh(t|\xi|(1-y))}{\cosh(t|\xi|)}|\xi|\mathcal{F}g + y\frac{\cosh(t|\xi|y)}{\cosh(t|\xi|)}\mathcal{F}h.\end{aligned}$$

The claim then follows from [18, Lemmata 2.2, 2.5, 2.6] combined with the operator-valued Fourier multiplier theorem [3, Theorem 6.2]. The fact that only the case $n = 2$ is considered in [18] is an immaterial difference, since the results involving n remain valid with only obvious modifications for $n > 2$. □

It is well known that

$$B = \Delta_x : \mathrm{dom}(B) \subset \mathrm{BUC}^\beta\big(\mathbb{R}^{n-1}\big) \to \mathrm{BUC}^\beta\big(\mathbb{R}^{n-1}\big) \tag{5.12}$$

generates a bounded analytic semigroup on $\mathrm{BUC}^\beta(\mathbb{R}, \mathrm{C}([0,1]))$ and it is easy the check by standard elliptic theory (see [20, Chapter 3] for instance) that

$$C = \partial_{yy} + ty\partial_y : \mathrm{dom}(C) \subset \mathrm{C}\big([0,1]\big) \to \mathrm{C}\big([0,1]\big) \tag{5.13}$$

(with boundary conditions) generates an exponentially decaying analytic semigroup on $\mathrm{BUC}^\beta(\mathbb{R}^{n-1}, \mathrm{C}([0,1]))$ for any fixed $t \in [0,T]$. In either case the missing variables can be considered as parameters. The sum $A(t) = B + \frac{1}{t^2}C$ then generates an exponentially decaying

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analytic semigroup on E_0 with constant domain of definition, provided $t > 0$. In order to verify that all conditions (2.27)–(2.29) are satisfied we still need to estimate

$$\big[A(t) - A(s)\big]\big(-A(\tau)\big)^{-q} = \left[\left(\frac{1}{s^2} - \frac{1}{t^2}\right)\partial_{yy} + \left(\frac{1}{s} - \frac{1}{t}\right)y\partial_y\right]\left[-\Delta_x - \frac{1}{\tau^2}\partial_{yy} - \frac{1}{\tau}y\partial_y\right]^{-q}$$

for $0 < \tau \leqslant s \leqslant t \leqslant T$ and $q = 1, p$. Since the estimate is only needed in the base space E_0, it causes no problem that

$$\big[-\tau^2\Delta_x - \partial_{yy} - \tau y\partial_y\big]^{-q}$$

looses its x-regularizing effect as $\tau \to 0$ provided ∂_{yy} (with boundary conditions) is invertible, as it is. This leads to the estimates

$$\big\|\big[A(t) - A(s)\big]\big(-A(\tau)\big)^{-q}\big\|_{\mathcal{L}(E_0)} \leqslant c(t-s)/t^{\frac{p-q}{p-1}}, \quad q = 1, p.$$

Theorem 4.4 can now be safely applied to obtain the following result.

Theorem 5.2. *Assume that* $g \in \mathrm{BUC}^{2+\beta}(\mathbb{R}^{n-1}, \mathbb{R})$ *and* $h \in \mathrm{BUC}^{1+\beta}(\mathbb{R}^{n-1}, \mathbb{R})$ *for* $\beta \in (0, 1)$. *Then problem* (5.5)–(5.7) *possesses a unique solution with*

$$u, \dot{u}, Au \in \mathrm{C}^{\alpha}_{\alpha}\big((0, T], \mathrm{BUC}^{\beta}\big(\mathbb{R}^{n-1}, \mathrm{C}\big([0, 1]\big)\big)\big).$$

It is given by

$$u(t) = \int_0^t U_A(t, \tau) f(\tau)\, d\tau + \mathcal{R}_D(t)g + t\mathcal{R}_N(t)h,$$

where U *is the evolution operator generated by the singular family* $A(t) = B + \frac{1}{t^2}C$ *with* B *and* C *as in* (5.12)–(5.13) *and the operators* $\mathcal{R}_D$ *and* $\mathcal{R}_N$ *are as defined in* (5.10)–(5.11).

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