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Elliptic and parabolic problems in unbounded domains

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We consider elliptic and parabolic problems in unbounded domains. We give general existence and regularity results in Besov spaces and semi-explicit representation formulas via operator-valued fundamental solutions which turn out to be a powerful tool to derive a series of qualitative results about the solutions. We give a sample of possible applications including asymptotic behavior in the large, singular perturbations, exact boundary conditions on artificial boundaries and validity of maximum principles.

1 Introduction

There are several reasons to consider elliptic and parabolic problems in unbounded domains. In particular, many physical situations are most suitably modeled as initial boundary value problems in unbounded domains. Mathematical considerations also often lead to consider problems of this kind. In this paper we propose a general semi-classical approach to the solution of elliptic and parabolic initial boundary value problems (BVPs and IBVPs, respectively) in unbounded domains of type $\mathbb{R}^n \times \Omega$ where $\Omega \subset \mathbb{R}^m$ is an open, bounded domain of $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and $m, n \in \mathbb{N}$. In recent years a lot of effort went into a theory for the sum of operators. The main results are the Da Prato-Grisvard and the Dore-Venni theorems. We refer to the original papers [16] and [9]. They were especially developed to deal with parabolic problems but could also be used to analyze the sum of two elliptic operators. Here we choose a different approach which is more in the spirit of [4]. It might be called abstract elliptic problems in Banach spaces in analogy to the concept of abstract evolution equations. It has the advantage of producing semi-classical formulas for the solution with far-reaching consequences for applications. More precisely we shall consider elliptic equations of the type

$$-\triangle u + Au \;=\; f \tag{1.1}$$

or parabolic equations of the form

$$\dot{u} - \triangle u + Au \;=\; f(t)\,, \quad t \;>\; 0\,, \quad u(0) \;=\; u_0 \tag{1.2}$$

where

$$A \in \mathcal{H}^-(E_1, E_0)$$

is the negative generator of an exponentially decaying C_0-semigroup e^{-tA} on the Banach space E_0 with domain of definition E_1 and f a vector-valued map taking values in E_0 and defined on $\mathbb{R}^n$ or $J \times \mathbb{R}^n$ for some time interval J with $0 \in J \subset \mathbb{R}^+$, respectively. In particular, the above class of problems covers the situation of BVPs and IBVPs in unbounded domains ot the type $\mathbb{R}^n \times \Omega$. It suffices to take for A the abstract operator associated to any uniformly strongly elliptic boundary value problem on the smooth domain Ω with appropriate spectral properties. Once a good solution theory for the abstract elliptic problem (1.1) is established, resolvent estimates

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and semigroup theory can be used to produce results for the abstract parabolic problem (1.2) and therefore for the corresponding class of IBVPs .

We now sketch how we are going to handle (1.1). An operator valued distribution

$$G \in \mathcal{S}'(\mathbb{R}^n, \mathcal{L}_s(E_0, E_1))$$

is called *semi-classical fundamental solution* for $-\triangle + A$ iff

$$(-\triangle + A)G \;=\; \delta \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n, \mathcal{L}_s(E_0))\ ,$$

that is, in the sense of vector-valued distributions. By $\mathcal{L}_s(E_0, E_1)$ and $\mathcal{L}_s(E_0)$ we denote the space of linear and continuous operators from E_0 to E_1 and on E_0, respectively, endowed with the strong topology. By $\mathcal{S}'(\mathbb{R}^n, E)$ we mean the space of E-valued Schwartz distributions. We refer to the next section for precise definitions.

If a distribution G is found with this property the solution u of (1.1) can be represented as the vector-valued convolution

$$u \;=\; G * f$$

with the datum f. An explicit formula for G is of interest in that it allows one to derive a series of important properties of the solution u as is well-known in the scalar case. The vector valued Fourier transform is a natural tool to use when trying to find fundamental solutions. Taking a partial Fourier transform with respect to $x \in \mathbb{R}^n$ of (1.1) yields

$$\big(\xi^2 + A\big)\hat{u} \;=\; \hat{f}\,.$$

The properties of the operator A imply that we can solve for $\hat{u}$ in the above expression to obtain

$$\hat{u} \;=\; \big(\xi^2 + A\big)^{-1}\hat{f} \;=:\; a(\xi)\hat{f}\,.$$

Now, operator-valued Fourier multiplier theorems have been established in recent years that allow one to analyze the symbol a and derive mapping properties of the associated pseudodifferential operator (ψDO)

$$a(D) \;=\; (-\triangle + A)^{-1} \;=\; \mathcal{F}^{-1} a \mathcal{F}$$

in a variety of function spaces. In order to deal with the problem of finding fundamental solutions for (1.1) we shall make use of available functional calculus techniques. The class of generators of analytic C_0-semigroups allows for a particularly powerful functional calculus based on the Dunford integral. Given a holomorphic function φ defined on some sector

$$\Sigma_\theta \;=\; \big\{\lambda \in \mathbb{C} \,\big|\, |\arg|(\lambda) \le \theta\big\} \quad \text{with} \quad \theta \in \big(\tfrac{\pi}{2}, \pi\big)$$

for which

$$|\lambda|^\delta\, \varphi(\lambda) \;\longrightarrow\; 0 \quad \text{as} \quad \lambda \;\longrightarrow\; \infty\,,$$

the integral

$$\frac{1}{2\pi i}\int_\Gamma \varphi(-\lambda)(\lambda + A)^{-1}\, d\lambda \;=\; \frac{1}{2\pi i}\int_{-\Gamma} \varphi(\lambda)(\lambda - A)^{-1}\, d\lambda$$

can be used to define a functional calculus for the operator A, that is an algebra homomorphism $\varphi \to \varphi(A)$ from

$$\Phi \;:=\; \big\{\varphi \in \mathcal{H}\left(\Sigma_\theta\right) \,\big|\, |\lambda|^\delta \varphi(\lambda) \to 0 \text{ as } \lambda \to \infty\big\}$$

into the space of linear and continuous operators $\mathcal{L}(E_0)$ on E_0 as long as A as the right spectral properties. This amounts essentially to asking for the operator A to be sectorial in a large enough sector. By means of this functional calculus and in perfect analogy to the scalar case it is possible to derive formulas for the fundamental solution G_n of the differential operator $-\triangle + A$ which of course depend on the dimension n. For $n = 1, 2, 3$ they are

$$G_1 \;=\; \sqrt{\frac{\pi}{2}}\, A^{-\frac{1}{2}}\, e^{-|x|\, A^{\frac{1}{2}}}\,, \tag{1.3}$$

$$G_2 \;=\; \frac{\pi}{2}\, K_0\big(|x|\, A^{\frac{1}{2}}\big)\,, \tag{1.4}$$

$$G_3 \;=\; \Big(\frac{\pi}{2}\Big)^{3/2}\, \frac{1}{|x|}\, e^{-|x|\, A^{\frac{1}{2}}}\,, \tag{1.5}$$

where K_0 is the modifed Bessel function of order 0. They satisfy $G_n \in \mathrm{L}_1(\mathbb{R}^n, \mathcal{L}(E_0))$ for any $n \in \mathbb{N}$ and corresponding formulas can be derived for $n \geq 4$ using the analogous relations for the associated scalar functions. As we shall see, the above formulas have a variety of different consequences and many important qualitative results can be proved with their help.

The paper is organized as follows. In the next section we formulate the problem and present our main results concerning regularity, existence and representation formulas for the solutions of (1.1) and (1.2). In Section 3 we demonstrate the power of our formulas considering several applications. We show how to handle uniformly elliptic boundary value problems in unbounded domains, analyze the asymptotic behavior of their solutions in the unbounded directions, consider singular perturbations, analyze the problem of imposing appropriate boundary conditions on artificial boundaries and prove the validity of maximum principles for them.

2 Formulation and main results

In this section we fill in the gaps of the above sketch as well as formulate and prove the main results. We begin with some necessary hypotheses and useful definitions.

Assume that the operator $-A$ is the generator of an analytic C_0-semigroup e^{-tA} on some Banach space E_0. We denote by E_1 its domain of definition endowed with the graph norm, so that $A \in \mathcal{L}(E_1, E_0)$, and assume the associated semigroup is exponentially decaying. More in general, given Banach spaces E_1 and E_0 with

$$E_1 \overset{d}{\hookrightarrow} E_0\,,$$

we define $\mathcal{H}^-(E_1, E_0)$ to be the collection of all negative generators with the above properties. The superscript means the E_1 is a dense subspace of E_0. We recall the basic inequality

$$\big\|(\lambda + A)^{-1}\big\|_{\mathcal{L}(E_0)} \;\leq\; c\,\frac{1}{1+|\lambda|}\,, \quad \lambda \in \Sigma_{\theta_A}\,, \tag{2.1}$$

which is valid in the set

$$S_{\theta_A} \;:=\; \{\lambda \in \mathbb{C} \mid |\arg|(\lambda) \leq \theta_A\} \cup \mathbb{B}(0, \varepsilon)$$

for some $\theta_A \in \big(\frac{\pi}{2}, \pi\big)$ and $\varepsilon > 0$ (cf. [15] or [12]). In particular $S_{\theta_A} \subset \rho(-A)$. The inequality can be used to introduce a functional calculus for the operator A and holomorphic functions $\varphi \in \Phi$ by setting

$$\frac{1}{2\pi i}\int_\Gamma \varphi(-\lambda)(\lambda + A)^{-1}\, d\lambda \;=\; \frac{1}{2\pi i}\int_{-\Gamma} \varphi(\lambda)(\lambda - A)^{-1}\, d\lambda \tag{2.2}$$

where the path of integration Γ is chosen in the set S_{θ_A} such as to connect $\infty e^{-i\tilde{\theta}}$ to $\infty e^{i\tilde{\theta}}$ for some $\tilde{\theta} < \theta_A$ avoiding the negative real axis and therefore the origin, too. The domain of holomorphy of the function φ should of course contain a sector-like neighborhood of the spectrum of $-A$ with $\theta + \theta_A > \pi$. Cauchy's theorem implies that $\varphi(A)$ is well-defined and that the map

$$\Phi \;\longrightarrow\; \mathcal{L}(E_0)\,, \quad \varphi \;\longmapsto\; \varphi(A)$$

is an algebra homomorphism (cf. [3, Lemma 4.6.1]). Later we shall see that a vast class of BVPs lead naturally to consider operators A with the above properties and therefore allow for the calculus.

Now we consider the *abstract elliptic problem*

$$-\triangle u + Au \;=\; f\,, \quad x \in \mathbb{R}^n\,,$$

for $A \in \mathcal{H}^-(E_1, E_0)$ and $f \in \mathcal{S}'(\mathbb{R}^n, E_0)$. The space of vector-valued Schwartz distributions was introduced in [17] and [18] where a general theory of vector-valued distributions is developed. We refer to [5] and [4] for a more readable and extensive treatment of the subject. Here we only mention some basic facts and definitions. The Schwartz space of vector-valued rapidly decreasing functions is defined as

$$\mathcal{S}(\mathbb{R}^n, E) \;=\; \left\{ \varphi \in \mathrm{C}^\infty(\mathbb{R}^n, E) \;\middle|\; \sup_{x \in \mathbb{R}^n,\ |\alpha| \le k} \big\| \big(\sqrt{1+x^2}\,\big)^m \partial^\alpha \varphi(x) \big\| < \infty \text{ for } m, k \in \mathbb{N} \right\}$$

and is endowed with its natural Fréchet topology generated by the standard family of seminorms. Then the corresponding space of tempered distributions is given by

$$\mathcal{S}'(\mathbb{R}^n, E) \;=\; \mathcal{L}\big(\mathcal{S}, E\big)$$

where $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the scalar space of rapidly decreasing functions. This is the standard procedure followed in order to generalize scalar distributions to the vector-valued case but needs care to be implemented properly and we therefore refer to [5] and [4] for the details.

A *regular distribution* T_u is a distribution which allows for an integral representation by means of a function $u \in \mathrm{L}_{1,\,loc}(\mathbb{R}^n, E)$, that is,

$$\langle T_u, \varphi \rangle \;=\; \int_{\mathbb{R}^n} u(x)\varphi(x)\,dx\,, \quad \varphi \in \mathcal{S}\,.$$

We then have

$$\mathcal{S}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathrm{L}_{1,loc}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n, E)\,.$$

Sobolev spaces can be introduced based on differentiability in the sense of distributions where the derivative of $u \in \mathcal{S}'(\mathbb{R}^n, E)$ is defined by

$$\langle \partial^\alpha u, \varphi \rangle \;=\; (-1)^\alpha \langle u, \partial^\alpha \varphi \rangle\,, \quad \varphi \in \mathcal{S}\,.$$

Given a Banach space E we denote the vector-valued Besov spaces defined in [4, Section 5] by

$$\mathrm{B}^s_{p,q}(\mathbb{R}^n, E)$$

for any $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. We also need the smaller spaces

$$b^s_{p,q}(\mathbb{R}^n, E)$$

defined as the closure of $\mathrm{B}^{s+1}_{p,q}(\mathbb{R}^n, E)$ in the topology induced by $\mathrm{B}^s_{p,q}(\mathbb{R}^n, E)$. We mention here special instances of Besov spaces we shall be particularly interested in when considering applications in Section 3.

$$\mathrm{B}^s_{\infty,\infty}(\mathbb{R}^n, E) \;=\; \mathrm{BUC}^s(\mathbb{R}^n, E) \quad \text{for} \quad s \in \mathbb{R}^+ \setminus \mathbb{N}\,, \tag{2.3}$$

$$b^s_{\infty,\infty}(\mathbb{R}^n, E) \;=\; buc^s(\mathbb{R}^n, E) \quad \text{for} \quad s \in \mathbb{R}^+ \setminus \mathbb{N}\,. \tag{2.4}$$

Eventhough Besov spaces are most easily introduced with the help of diadic decompositions and pseudodifferential operators as in [4], they also allow for intrinsic definitions of their norms (see [20] for an exhaustive treatment of the subject). In particular

$$\|u\|_{\mathrm{BUC}^s(\mathbb{R}^n,E)} \;=\; \sup_{|\alpha| \le [s],\, x \in \mathbb{R}^n} \|\partial^\alpha u(x)\| + \sup_{|\alpha| = [s],\, x \ne y} \frac{\|\partial^\alpha u(x) - \partial^\alpha u(y)\|}{|x-y|^{s-[s]}} \tag{2.5}$$

can be used as a norm on $\mathrm{BUC}^s(\mathbb{R}^n,E)$. Hereby we denoted the integer part of s by the standard notation $[s]$. The spaces $buc^s(\mathbb{R}^n,E)$ can then be recovered as the closure of $\mathrm{BUC}^{[s]+1}$ in $\mathrm{BUC}^s(\mathbb{R}^n,E)$. We refer to [14, Section 0.2] for this fact.

The vector-valued Fourier transform $\mathcal{F}$ is defined through

$$(\mathcal{F}u)(x) \;=\; \hat{u}(x) \;=\; (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx\,, \quad x\in\mathbb{R}^n\,.$$

It enjoys the standard properties

$$\mathcal{F}\in\mathcal{L}(\mathrm{L}_1(\mathbb{R}^n,E),\mathrm{C}_0(\mathbb{R}^n,E))\cap\mathcal{L}is(\mathrm{L}_2(\mathbb{R}^n,E))\cap\mathcal{L}is(\mathcal{S}(\mathbb{R}^n,E))\cap\mathcal{L}is(\mathcal{S}'(\mathbb{R}^n,E))\;.$$

The letters "is" are appended to mean that the operator is actually an isomorphism on the given space. We also have that

$$\mathcal{F}^{-1}u \;=\; \overset{\nu}{\hat{u}} \;=\; \widehat{\overset{\nu}{u}}$$

for any $u\in\mathcal{S}'(\mathbb{R}^n,E)$ whereby

$$\langle\overset{\nu}{u},\varphi\rangle \;=\; \langle u,\overset{\nu}{\varphi}\rangle\,,\quad \varphi\in\mathcal{S}$$

and $\overset{\nu}{\varphi}(x)=\varphi(-x)$ for $x\in\mathbb{R}^n$. Many of the properties of scalar Fourier transform carry over to the vector-valued case (cf. [4]). Given Banach spaces E and F we introduce the multiplier space

$$\begin{aligned}\mathcal{O}_M\big(\mathbb{R}^n,\mathcal{L}(E,F)\big) \;=\; \big\{&\varphi\in\mathrm{C}^\infty(\mathbb{R}^n,\mathcal{L}(E,F))\mid \|\partial^\alpha\varphi(x)\|_{\mathcal{L}(E,F)}\le c_\alpha\,(1+|x|)^{m_\alpha}\\ &\text{for any } \alpha\in\mathbb{N}^n \text{ and some } c_\alpha>0,\ m_\alpha\in\mathbb{N}\big\}\end{aligned}$$

which is made into a locally convex space by the obvious choice of seminorms. We observe that

$$\mathcal{O}_M(\mathbb{R}^n,\mathcal{L}(E,F)) \;\overset{d}{\hookrightarrow}\; \mathcal{S}'(\mathbb{R}^n,\mathcal{L}(E,F))$$

and that

$$\mathcal{O}_M(\mathbb{R}^n,\mathcal{L}(E,F))\times\mathcal{S}'(\mathbb{R}^n,E)\;\longrightarrow\;\mathcal{S}'(\mathbb{R}^n,F)\,,\quad (\varphi,u)\;\longmapsto\;\varphi u$$

is a well-defined bilinear and continuous map (cf. [5]). Given

$$a\in\mathcal{O}_M(\mathbb{R}^n,\mathcal{L}(E,F))$$

we can consider the ψDO

$$a(D) \;=\; \mathcal{F}^{-1}a\mathcal{F}\in\mathcal{L}(\mathcal{S}(\mathbb{R}^n,E),\mathcal{S}(\mathbb{R}^n,F))\cap\mathcal{L}(\mathcal{S}'(\mathbb{R}^n,E),\mathcal{S}'(\mathbb{R}^n,F))$$

associated to it. The goal of Fourier multiplier theorems is to find sufficient conditions for a symbol $a\in\mathcal{S}'(\mathbb{R}^n,\mathcal{L}(E,F))$ to give rise to a pseudo-differential operator $a(D)$ for which

$$a(D)\in\mathcal{L}(\mathcal{F}(\mathbb{R}^n,E),\mathcal{F}(\mathbb{R}^n,F))$$

for a wide choice of function spaces $\mathcal{F}$ such that

$$\mathcal{S}(\mathbb{R}^n,X)\;\hookrightarrow\;\mathcal{F}(\mathbb{R}^n,X)\;\hookrightarrow\;\mathcal{S}'(\mathbb{R}^n,X)$$

for $X=E,F$. The following general multiplier theorem was proven by Amann in [4] and answers the questions in the case of Besov spaces.

Theorem 2.1 *Let $s\in\mathbb{R}$ and $p,q\in[1,\infty]$. Suppose that $\mathcal{B}\in\{\mathrm{B},b\}$, that $m\in\mathbb{R}$ and that*

$$\begin{aligned}a\in\mathrm{S}^m(\mathbb{R}^n,\mathcal{L}(E,F)) \;=\; \big\{&a\in\mathrm{C}^{n+1}(\mathbb{R}^n\setminus\{0\},\mathcal{L}(E,F))\mid\|\partial^\alpha a(\xi)\|\le c(1+|\xi|)^{m-|\alpha|}\,,\\ &\xi\in\mathbb{R}^n\setminus\{0\}\,,\ |\alpha|\le n+1\big\}\,.\end{aligned}$$

Then $\mathrm{S}^m(\mathbb{R}^n,\mathcal{L}(E,F))$ is a Banach space and

$$[a\mapsto a(D)]\in\mathcal{L}\left(\mathrm{S}^m(\mathbb{R}^n,\mathcal{L}(E,F)),\mathcal{L}\big(\mathcal{B}^{s+m}_{p,q}(\mathbb{R}^n,E),\mathcal{B}^s_{p,q}(\mathbb{R}^n,E)\big)\right)\,. \tag{2.6}$$

Definition 2.2 A distribution $G \in \mathcal{S}'(\mathbb{R}^n, \mathcal{L}_s(E_0, E_1))$ is called *semi-classical Green function* or *fundamental solution for* $-\triangle + A$ iff

$$-\triangle G + AG \;=\; \delta \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n, \mathcal{L}_s(E_0))\,.$$

If a Green function exists then of course

$$G * f \;=\; \int_{\mathbb{R}^n} G(x - \tilde{x}) f(\tilde{x})\, d\tilde{x}$$

represents a solution of (1.1) whenever a meaning can be attached to the right-hand side. We observe that the convolution might need to be understood in the sense of distributions. As in the scalar case Fourier transform is a powerful tool to compute Green functions. Taking a Fourier transform with respect to $x \in \mathbb{R}^n$ we are led to the representation

$$\hat{u} \;=\; \bigl(\xi^2 + A\bigr)^{-1} \hat{f}$$

in Fourier space. The invertiblity of $\xi^2 + A$ for any $\xi \in \mathbb{R}^n$ follows from the hypothesis $A \in \mathcal{H}^-(E_1, E_0)$. Using the above vector-valued Fourier multiplier theorem we can now obtain regularity results for (1.1).

Theorem 2.3 *Assume that* $A \in \mathcal{H}^-(E_1, E_0)$. *Then*

$$(-\triangle + A)^{-1} \;=\; \mathcal{F}(\xi^2 + A)^{-1}\mathcal{F} \;\; \in \mathcal{L}is\bigl(\mathcal{B}^s_{p,q}(\mathbb{R}^n, E_0), \mathcal{B}^{s+2}_{p,q}(\mathbb{R}^n, E_0) \cap \mathcal{B}^s_{p,q}(\mathbb{R}^n, E_1)\bigr) \tag{2.7}$$

for any $\mathcal{B} \in \{\mathrm{B}, b\}$, *any* $s \in \mathbb{R}$ *and any* $p, q \in [1, \infty]$.

P r o o f. The proof is an application of 2.1. The estimate

$$\bigl\|(\xi^2 + A)^{-1}\bigr\|_{\mathcal{L}(E_0, E_j)} \;\le\; c\, \frac{1}{(1 + \xi^2)^{1-j}}$$

for $j = 0, 1$ which is valid on the assumption that $A \in \mathcal{H}^-(E_1, E_0)$ and the equality

$$\partial_j \bigl(\xi^2 + A\bigr)^{-1} \;=\; 2\xi_j \bigl(\xi^2 + A\bigr)^{-2}, \quad j \;=\; 1, \ldots, n$$

entail by induction that

$$\bigl(\xi^2 + A\bigr)^{-1} \in \mathcal{S}^{-2}(\mathbb{R}^n, \mathcal{L}(E_0)) \cap \mathcal{S}^0(\mathbb{R}^n, \mathcal{L}(E_0, E_1))\,.$$

The claim follows. □

Remark 2.4 The deep Fourier multiplier theorem 2.1 is not necessary to see that the weaker integrabilty property

$$\bigl(\xi^2 + A\bigr)^{-1} \in \mathcal{F}\,\mathrm{L}_1(\mathbb{R}^n, \mathcal{L}(E_0, E_\alpha))$$

is valid for $\alpha \in (0, 1)$ and E_α an interpolation space between E_0 and E_1 of exponent α. We only need to use a weaker result which holds for "slightly" decaying symbols and is proven in [4, Corollary 4.4] combined with interpolation inequality

$$\|x\|_{E_\alpha} \;\le\; c\, \|x\|^{1-\alpha}_{E_0}\, \|x\|^{\alpha}_{E_1}$$

which is valid for some $c > 0$ and estimate (2.1) for A.

Now we shall use Dunford functional calculus for generators of analytic C_0-semigroups to prove the following

Theorem 2.5 *Assume that $A \in \mathcal{H}^-(E_1, E_0)$. Then there exist fundamental solutions G_n for $-\triangle + A$ which satisfy*

$$G_n \in \mathrm{L}_1(\mathbb{R}^n, \mathcal{L}(E_0, E_\alpha))$$

for any $\alpha \in [0,1)$ and are given by

$$G_1 \;=\; \sqrt{\frac{\pi}{2}}\, A^{-\frac{1}{2}}\, e^{-|x|\,A^{\frac{1}{2}}}\,,$$

by

$$G_n \;=\; \Big(\frac{\pi}{2}\Big)^{\frac{n}{2}}\, \frac{e^{-|x|\,A^{\frac{1}{2}}}}{|x|^{n-2}} \qquad \text{if } n \geq 3 \text{ is odd}$$

and by

$$G_n \;=\; \Big(\frac{\pi}{2}\Big)^{\frac{n}{2}}\, K_0\big(|x|\,A^{\frac{1}{2}}\big) \quad \text{if } n \text{ is even}.$$

P r o o f. We certainly know that, if G_n exists, then its Fourier transform must be given by $\big(\xi^2 + A\big)^{-1}$. It is however easier to give a direct proof than to compute the corresponding inverse Fourier transform. Let the functions g_n be defined through

$$g_1(x,\lambda) \;=\; \sqrt{\frac{\pi}{2}}\,\lambda^{-1}\, e^{-|x|\,\lambda}\,,$$

$$g_{2k}(x,\lambda) \;=\; \frac{\pi}{2}\, K_0(|x|\,\lambda)\,,$$

$$g_{2k+1}(x,\lambda) \;=\; \Big(\frac{\pi}{2}\Big)^{\frac{3}{2}}\, \frac{e^{-|x|\,\lambda}}{|x|^{n-2}}\,,$$

for $k \in \mathbb{N}\setminus\{0\}$. We observe that they are holomorphic for $\lambda \in \mathbb{C}\setminus\mathbb{R}^-$ and that g_n is even exponentially decaying for $\Re e\,\lambda > 0$ for any $n \in \mathbb{N}$. Since the complex square root function is also holomorphic in the same set g_n can be used to define G_n by means of Dunford calculus as

$$G_n(x,A) \;=\; \frac{1}{2\pi i}\int_\Gamma g_n\big(x,\sqrt{\lambda}\big)(\lambda - A)^{-1}\, d\lambda$$

where the path of integration is chosen as to connect $\infty e^{-i\tilde\theta}$ to $\infty e^{i\tilde\theta}$ for some $\tilde\theta < \theta_A$ avoiding the negative real axis. This particularly convenient situation, as far as convergence of the integral is concerned, is due to the fact that g_n are actually functions of $\sqrt{\lambda}$.

We observe that the fractional power $A^{\frac{1}{2}}$ can be defined as the inverse of $A^{-\frac{1}{2}}$, which in its turn can be defined by Dunford calculus. It is thus a consequence of [3, Proposition 4.6.10] that $-A^{\frac{1}{2}}$ also generates an analytic C_0-semigroups on E_0.

It follows that G_{2k+1}, $k \in \mathbb{N}$, can be understood in the sense of semigroups. In these terms we can also understand the asymptotic behavior of G_{2k}, $k \in \mathbb{N}\setminus\{0\}$, for large $x \in \mathbb{R}^2$. Semigroup theory and Dunford calculus thus also give a way of understanding the singularity in $x = 0$. Now we can use the formula

$$G_n(x,A) \;=\; \frac{1}{2\pi i}\int_\Gamma g_n\big(x,\sqrt{\lambda}\big)(\lambda - A)^{-1}\, d\lambda$$

and the fact that

$$(\lambda - \triangle)g_n \;=\; \delta$$

to see that

$$(-\triangle + A)\,A^{-\epsilon}G_n\big(x, A^{\frac{1}{2}}\big) \;=\; A^{-\epsilon}\delta$$

for $\epsilon \in (0,1)$. We make use of the operator $A^{-\epsilon}$ to ensure the convergence of the integrals. Alternatively, we could directly compute the integral in the strong sense. Since A^{-t} is an analytic C_0-semigroup on E_0, as follows from [3, Theorem 4.6.4], we now have the desired strong convergence

$$A^{-\epsilon} \longrightarrow \mathrm{id}_{\mathcal{L}_s(E_0)}, \quad \epsilon \longrightarrow 0$$

which concludes the proof. □

Remarks 2.6 (a) We observe that, whenever n is odd, semigroup theory can now be use directly on the Green function to produce a whole series of estimates for $(-\triangle + A)^{-1}$.

(b) In particular, information about exponential and Bessel functions combined with Dunford calculus or semigroup theory can be used to understand the sigularity in the origin for the Green function.

Main theorem 2.5 has also a lot of nice qualitative consequences as we shall see in the applications of the last section. Here we only mention a particular application which we can formulate in an abstract way and will be used in the derivation of a maximum principle for BVPs.

Assume that the space E_0 is an ordered Banach space and that $\mathcal{P} \subset E_0$ is the cone of positive elements. An operator $T \in \mathcal{L}(E_0)$ is called positive iff

$$T(\mathcal{P}) \subset \mathcal{P}.$$

The notation $T \in \mathcal{L}^+(E_0)$ indicates that the operator is positive. We refer to [3, Section II.6] for a more detailed description of this topic.

Definition 2.7 An unbounded operator A on E_0 is called *resolvent positive* iff $(t + A)^{-1}$ exists for any $t \in (0,\infty)$ and is positive for any such t.

We have now the following

Theorem 2.8 *Assume that $A \in \mathcal{H}^-(E_1,E_0)$ is resolvent positive. Then the Green function G_n defines a positive operator, that is,*

$$G_n * f$$

is positive, whenever f is positive or, equivalently, whenever $f(x) \in \mathcal{P}$ for almost any $x \in \mathbb{R}^n$. If n is odd we even have that

$$G_n(x)(\mathcal{P}) \subset \mathcal{P}$$

for any $x \in \mathbb{R}^n$.

P r o o f. We divide the proof in two parts. We begin with n odd. If an operator A is resolvent positive, then so is $A^{\frac{1}{2}}$. This can be seen by contracting the integration path Γ to a double cover of $\mathbb{R}$ to obtain

$$\big(t + A^{\frac{1}{2}}\big)^{-1} = \frac{1}{\pi}\int_0^\infty \frac{r^{1/2}}{t^2 + r}\,(r + A)^{-1}\,dr\,.$$

It thus follows from [3, Theorem 6.3.2] that the semigroup generated by $-A^{\frac{1}{2}}$ also enjoys the same property, that is,

$$e^{-tA^{\frac{1}{2}}}\mathcal{P} \subset \mathcal{P} \quad \text{for any} \quad t \in \mathbb{R}^+ \tag{2.8}$$

and the assertion easily follows from the representation formulas given in 2.5. Now, let us consider the case when n is even, which is a little more involved. In view of the properties of the functions g_n mentioned in the proof of main theorem 2.5 we can contract the path of integration Γ to $\mathbb{R}^-$ to obtain

$$g_n(x,A) = \frac{1}{2\pi i}\int_0^\infty \big(g_n\big(-i\sqrt{\tau}\,\big) - g_n\big(i\sqrt{\tau}\,\big)\big)(\tau + A)^{-1}\,d\tau\,. \tag{2.9}$$

Now it can be seen that $\frac{1}{i}\big(g_n\big(-i\sqrt{\tau}\big) - g_n\big(i\sqrt{\tau}\big)\big)$ is a real-valued function of $\tau \in (0,\infty)$. Unfortunately it is oscillating about zero, but it is initially positive and has a monotonically narrowing envelop as seen from the estimate

$$\big|\big(g_n\big(-i\sqrt{\tau}\big) - g_n\big(i\sqrt{\tau}\big)\big)\big| \;\le\; \frac{5}{1+\tau^{1/4}}\,, \quad \tau \;>\; 0\,,$$

which also implies the existence of the integral (2.9). If $(\tau + A)^{-1}$ was monotonically decreasing the assertion would follow. Let u be any positive vector in E_0, then

$$(\tau + A)^{-1}u - (\sigma + A)^{-1}u \;=\; -\int_\sigma^\tau (\rho + A)^{-2}u\, d\rho\,.$$

Of course $(\rho + A)^{-2}$ is also positive and thus the above difference is negative whenever $\tau > \sigma$ which proves that

$$\big[\tau \mapsto (\tau + A)^{-1}\big]\,, \quad (0,\infty) \;\longrightarrow\; \mathcal{L}^+(E_0)$$

is monotonically decreasing and the proof is complete. □

Now we turn our attention to parabolic problems. Substituing $\big(\xi^2 + A\big)^{-1}$ with $\big(\lambda + \xi^2 + A\big)^{-1} = \big(\xi^2 + (\lambda + A)\big)^{-1}$ and using multiplier theorem 2.1 to derive resolvent estimates for $-\triangle + A$ it is easy to prove the following generation theorem.

Theorem 2.9 *Let the hypotheses of main Theorem* 2.3 *be satisfied. Then*

$$-\triangle + A \in \mathcal{H}\big(buc^{s+2}(\mathbb{R}^n, E_0) \cap buc^s(\mathbb{R}^n, E_1)\,,\, buc^s(\mathbb{R}^n, E_0)\big) \tag{2.10}$$

for any $s \in \mathbb{R}^+ \setminus \mathbb{N}$.

Remark 2.10 The reason for the choice of the "little" Besov spaces buc^s in the above theorem is due to the fact that for an operator A to belong to the class $\mathcal{H}(F_1, F_0)$ means in particular that the semigroup e^{-tA} be strongly continuous in the origin. This is equivalent to F_1 being dense in F_0 which is not always true for general Besov spaces. Take the spaces $\mathrm{BUC}^s(\mathbb{R}^n)$ as an example. It is, however, true for bus^s spaces and other spaces as well. We refer to [4] for the more general case.

Proof. The necessary resolvent estimates (cf. [3, Section I.1.2]) follow from the properties of the operator A, the symbol estimates for $\big(\lambda + \xi^2 + A\big)^{-1}$ and Theorem 2.1 together with the standard characterization of generators of analytic semigroups as found in, e.g., [15]. □

The above theorem implies, in particular, that we can make use of the well developed theory of analytic C_0-semigroups to prove existence and regularity theorems for the semi-abstract parabolic equation

$$u_t - \triangle u + Au \;=\; f(t)\,, \quad t \;>\; 0\,, \quad u(0) \;=\; u_0\,.$$

We refer to standard books about semigroup theory, such as [15], [12] and [10], for typical applications to linear and semilinear evolution equations. We also refer to the more exhaustive treatment by Amann in [3] which sets the basis for treating the quasilinear and the fully nonlinear cases. We conlude this section with some general remarks.

Remarks 2.11 (a) The techniques employed are by no means restricted to second order operators and can therefore be extended to higher order operators.

(b) Here we only considered constant coefficients in the variable x. One can now try to handle the case of non constant coefficients by localization techniques. Of course explicit representation formulas like (1.3) would not be valid anymore.

(c) The above results can be used in the analysis of nonlinear problems of semilinear type.

Some of the above topics are going to be included in the forthcoming volumes II and III of [5].

3 Applications

This section is devoted to a series of applications of the results obtained in the previous section. They are chosen as to give a glance at the variety of possible applications one can envision and designed to be very simple and clear.

3.1 Boundary value problems

Let $\Omega \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) be an open, bounded domain with smooth boundary $\partial\Omega$ and let

$$\mathcal{A}(y, \partial) \;=\; \operatorname{div}\big(\Lambda\nabla u + bu\big) + (c|\nabla u) + du\,, \tag{3.1}$$

$$\mathcal{B}(y, \partial) \;=\; (1 - \delta)u + \delta\big(\partial_{\nu_\Lambda} u + [(\gamma_\partial b|\nu) + \beta_0]\gamma_\partial u\big)\,. \tag{3.2}$$

Assume for simplicity that all data are smooth and that $\delta \in \mathrm{C}(\partial\Omega, \{0, 1\})$. Then $\partial\Omega$ has components $\Gamma_j = \delta^{-1}(j)\,,\; j = 0, 1$. The vector ν is the unit outward normal to $\partial\Omega$, $\partial_{\nu_\Lambda} u = (\nabla u|\Lambda\nu)$ is the conormal derivative of u and γ_∂ is the trace operator. Also assume that there exists $\underline{\alpha} > 0$ with

$$(\Lambda(y)\eta|\eta) \;\geq\; \underline{\alpha}\,|\eta|^2 \quad \text{for any} \quad \eta \in \mathbb{R}^m \;\text{ and }\; y \in \Omega\,.$$

To $(\mathcal{A}, \mathcal{B})$ we associate the uniformly normally elliptic boundary value problem

$$-\triangle u + \mathcal{A}u \;=\; f\,, \quad (x, y) \in \mathbb{R}^n \times \Omega\,, \tag{3.3}$$

$$\mathcal{B}u \;=\; g\,, \quad (x, y) \in \mathbb{R}^n \times \partial\Omega\,. \tag{3.4}$$

It is a standard procedure to construct from $(\mathcal{A}, \mathcal{B})$ an abstract operator A on $\mathrm{L}_p(\Omega)$ for $p \in (1, \infty)$ with

$$\operatorname{dom}(A) \;=\; W^2_{p,\mathcal{B}}(\Omega) \;=\; \big\{u \in W^2_p(\Omega) \mid \mathcal{B}u = 0\big\}$$

and $Au = \mathcal{A}u$ for $u \in \operatorname{dom}(A)$. A proof of

$$A \in \mathcal{H}\big(W^2_{p,\mathcal{B}}(\Omega), \mathrm{L}_p(\Omega)\big)$$

can be found in [2], where it is also proved that the associated analytic C_0-semigroup is exponentially decaying if $\delta \neq 1$ or, if $\delta = 1$ but $\beta_0 \neq 0$, which make sure that $0 \in \rho(A)$. Now, if $g \neq 0$ in (3.4) then one can use trace theorems to find

$$w \in W^2_p(\Omega) \quad \text{with} \quad \mathcal{B}w \;=\; g\,, \tag{3.5}$$

which is possible on the assumption that

$$g \in W^{2-1/p}_p(\Gamma_0) \times W^{1-1/p}_p(\Gamma_1)$$

(see for instance [1] and [19]). Looking for a solution in the form $u = v + w$ we see that v satisfies a boundary value problem with homogeneous boundary condition and modified right-hand side $\tilde{f} = f - \mathcal{A}w \in \mathrm{L}_p(\Omega)$. Using this facts and main theorems 2.3 and 2.5 we can prove

Theorem 3.1 *Let the above assumptions be satisifed and $s \in \mathbb{R}^+ \setminus \mathbb{N}$. Then the boundary value problem* (3.3)–(3.4) *possesses for any*

$$(f, g) \in \mathrm{BUC}^s\big(\mathbb{R}^n, \mathrm{L}_p(\Omega)\big) \times \mathrm{BUC}^{s+2}\big(\mathbb{R}^n, W^{2-1/p}_p(\Gamma_0) \times W^{1-1/p}_p(\Gamma_1)\big)$$

a unique solution u with

$$u \in \mathrm{BUC}^{s+2}(\mathbb{R}^n, \mathrm{L}_p(\Omega)) \cap \mathrm{BUC}^s\left(\mathbb{R}^n, W^2_p(\Omega)\right).$$

The solution is given by

$$u(x) \;=\; \int_{\mathbb{R}^n} G_n(x - \tilde{x})\big(f - (-\triangle + \mathcal{A})w\big)(\tilde{x})\, d\tilde{x} + w$$

where

$$G_n(x) = \begin{cases} \sqrt{\frac{\pi}{2}}\, A^{-\frac{1}{2}} e^{-|x|\, A^{\frac{1}{2}}}\,, & \text{if} \quad n = 1\,, \\ \left(\sqrt{\frac{\pi}{2}}\right)^{n/2} K_0\big(|x|\, A^{\frac{1}{2}}\big)\,, & \text{if} \quad n \text{ is even}\,, \\ \left(\sqrt{\frac{\pi}{2}}\right)^{n/2} \frac{1}{|x|^{n-2}}\, e^{-|x|\, A^{\frac{1}{2}}}\,, & \text{otherwise}\,, \end{cases}$$

and w is chosen as in (3.5).

P r o o f. The proof follows from main theorems 2.3 and 2.5 along with (2.3). □

Using main theorem 3 we derive in the same manner the following

Theorem 3.2 *Let $\rho \in (0,1)$ and assume that*

$$\begin{aligned} & f \in \mathrm{C}^\rho\big(J, buc^s\big(\mathbb{R}^n, \mathrm{L}_p(\Omega)\big)\big)\ , \\ & g \in \mathrm{C}^{1+\rho}\left(J, buc^s\big(\mathbb{R}^n, W_p^{2-1/p}(\Gamma_0) \times W_p^{1-1/p}(\Gamma_1)\big)\right) \\ & \quad \cap\, \mathrm{C}^\rho\left(J, buc^{s+2}\big(\mathbb{R}^n, W_p^{2-1/p}(\Gamma_0) \times W_p^{1-1/p}(\Gamma_1)\big)\right)\ , \\ & u_0 \in buc^s\big(\mathbb{R}^n, \mathrm{L}_p(\Omega)\big)\,. \end{aligned}$$

Then there exists a unique solution u of

$$u_t - \triangle u + \mathcal{A}u = f\,, \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^n \times \Omega\,, \tag{3.6}$$

$$\mathcal{B}u = g\,, \quad (t,x,y) \in (0,\infty) \times \mathbb{R}^n \times \partial\Omega\,, \tag{3.7}$$

$$u(0) = u_0\,, \tag{3.8}$$

with

$$u \in \mathrm{C}^\rho\big(J\setminus\{0\}, buc^{s+2}\big(\mathbb{R}^n, \mathrm{L}_p(\Omega)\big) \cap buc^s\big(\mathbb{R}^n, W_p^2(\Omega)\big)\big) \cap \mathrm{C}^{1+\rho}\left(J\setminus\{0\}, buc^s\big(\mathbb{R}^n, \mathrm{L}_p(\Omega)\big)\right)\,.$$

P r o o f. The proof is an immediate consequence of main theorem 3 and of [3, Theorem 1.2.1] □

3.2 Asymptotic behavior of solutions

In this subsection we are interested in the behavior of the solution of BVP in the unbounded domain $\mathbb{R}^n \times \Omega$ for large $x \in \mathbb{R}^n$. We shall only consider a simple situation.

Remark 3.3 Several authors have considered the problem of comparing the asymptotic behavior of a parabolic problem with the one of the corresponding elliptic regularization. This is the case when $n = 1$ and the x-variable is considered as time. See [7], [8] and the references therein. Eventhough we consider here the whole space case it is not difficult to derive analogous representation formulas in the half space case in a similar manner as one would do in the scalar case.

On the assumption that $g = 0$ and that f has an asymptotic behavior given by

$$f(x,\cdot) \longrightarrow f_\infty \ \text{ in } \ \mathrm{L}_p(\Omega) \ \text{ as } \ |x| \longrightarrow \infty$$

that is, f has an homogeneous limit. We shall see that u has also an asymptotic behavior, as a consequence, which is determined by the solution of the BVP

$$\mathcal{A}u = f_\infty \quad \text{in} \quad \Omega\,, \tag{3.9}$$

$$\mathcal{B}u = 0 \quad \text{on} \quad \partial\Omega\,. \tag{3.10}$$

This might be rephrased by saying that the operators in the sum $-\triangle + A$ asymptotically decouple. We formulate the result as follows

Theorem 3.4 *Assume that f has the given asymptotic behavior. Then the solution u of* (3.3)–(3.4) *satisfies*

$$u(x,\cdot) \longrightarrow u_\infty \text{ in } W_p^2(\Omega) \text{ as } |x| \longrightarrow \infty$$

where u_∞ is determined as the solution of the BVP (3.9)–(3.10)*.*

P r o o f. On the given assumptions the solution of (3.3)–(3.4) is given by

$$u(x,\cdot) = \int_{\mathbb{R}^n} G_n(\tilde{x}) f(x-\tilde{x},\cdot)\, d\tilde{x}\,.$$

Now, the above integrand converges pointwise towards $G(\tilde{x})f_\infty$. Since the semi-classical Green function is strongly integrable we conclude with the Lebesgue theorem that

$$u(x,\cdot) \longrightarrow \int_{\mathbb{R}^n} G_n(\tilde{x}) f_\infty\, d\tilde{x} \quad \text{as} \quad |x| \longrightarrow \infty$$

but $\big(\mathcal{F}G_n\big)(0) = A^{-1}$, thus

$$u_\infty = A^{-1} f_\infty$$

and the theorem is proved. □

3.3 Maximum principle

In applications, especially to nonlinear equations, positivity properties of solution operators are often of great help. We refer for instance to [6] where the authors consider, among other things, the question about the validity of a maximum principle for the particular case of the Laplacian $-\triangle_x - \triangle_y$. With the use of Theorem 2.8 we are able to derive a maximum principle for a wide class of problems.

Theorem 3.5 *Assume that the BVP $(\mathcal{A},\mathcal{B})$ in Ω allows for a maximum principle, that is, that $\mathcal{A}u \geq 0$ and $\mathcal{B}u \geq 0$ imply that the solution $u \geq 0$. Then $(-\triangle + \mathcal{A},\mathcal{B})$ in $\mathbb{R}^n \times \Omega$ also has a maximum principle.*

P r o o f. On the above assumptions it is not difficult to see that the operator A associated to $(\mathcal{A},\mathcal{B})$ is a resolvent positive operator in $\mathrm{L}_p(\Omega)$. The assertion is thus a consequence of Theorem 2.8. □

Remark 3.6 Observe that we are actually assuming that the BVP $(\mathcal{A},\mathcal{B})$ has local boundary conditions (see the definitions at the beginning of the section). It was in fact shown in [13] that resolvent positivity does not follow from the validity of a maximum principle in the case non separated boundary conditions. In that case we would have to assume resolvent positivity the operator A associated to $(\mathcal{A},\mathcal{B})$ directly.

3.4 Singular perturbations

Consider the BVP

$$-\epsilon\triangle u + \mathcal{A}u = f\,, \quad (x,y)\in\mathbb{R}^n\times\Omega\,, \tag{3.11}$$
$$\mathcal{B}u = 0\,, \quad (x,y)\in\mathbb{R}^n\times\partial\Omega\,, \tag{3.12}$$

for positive $\epsilon > 0$ and some $f \in \mathrm{BUC}^s(\mathbb{R}^n,\mathrm{L}_p(\Omega))$. We are interested in the behavior of its solution as $\epsilon \to 0$.

Remark 3.7 Perturbation problems of the above are of interest for many different reasons. Multidimensional models characterized by small aspect ratios which also neglect edge effects naturally lead to such problems. We also mention here a connection to the dynamical approach in the analysis of elliptic problems in cylindrical domains. See in particular the already mentioned [7] and [8].

Theorem 3.8 *Let u_ϵ be solution of the above BVP for any given $\epsilon \in (0,\infty)$. Then, as ϵ tends to zero, the solution u_ϵ converges towards the function u_0 given as the solution of the following parameter dependent BVP*

$$\mathcal{A}v = f\,, \quad y\in\Omega\,, \tag{3.13}$$
$$\mathcal{B}v = 0\,, \quad y\in\partial\Omega\,, \tag{3.14}$$

in the topology of $\mathrm{BUC}^{s+2}(\mathbb{R}^n,\mathrm{L}_p(\Omega)) \cap \mathrm{BUC}^s\big(\mathbb{R}^n, W_p^2(\Omega)\big)$.

Proof. The proof is an easy consequence of the fact that

$$1/\sqrt{\epsilon}\,\sigma_{1/\sqrt{\epsilon}} G_n \;=\; \mathcal{F}^{-1}\big(\epsilon\xi^2 + A\big)^{-1}\mathcal{F} \;\longrightarrow\; A^{-1}\delta \quad \text{as} \quad \epsilon \longrightarrow 0$$

in the topolgy of $\mathcal{S}'\big(\mathbb{R}^n, \mathcal{L}_s(E_0, E_1)\big)$. Where the dilation operator σ_τ is defined by $(\sigma_\tau g)(x) = g(\tau x)$ for $x \in \mathbb{R}^n$ and any function g. It suffices to observe that

$$u_\epsilon \;=\; 1/\sqrt{\epsilon}\,\sigma_{1/\sqrt{\epsilon}} G_n * f$$

to conclude that the claimed convergence indeed takes place. □

3.5 Artificial boundary conditions

In this last subsection we give an example of a problem which is very often encountered in the numerical treatment of problems in unbounded domains. That is, the problem of finding appropriate boundary conditions to be imposed on artificial boundary introduced to make the computational domain finite. There is a vast literature about this problem (see [11], for instance, and the many references cited therein) and it is well-known that exact artificial boundary conditions can be formulated in terms of the Dirichlet-to-Neumann (DtN) operator. By means of the semi-classical Green functions we constructed we are able to make this procedure completely explicit. Consider the BVP

$$-\triangle u \;=\; f\,, \quad (x,y) \in \mathbb{R}\times\mathbb{B}^2(0,1)\,, \tag{3.15}$$

$$u \;=\; 0\,, \quad (x,y) \in \mathbb{R}\times\mathbb{S}^1\,, \tag{3.16}$$

in the infinite cylinder $\mathbb{R}\times\mathbb{B}^2 \subset \mathbb{R}^3$ with base $\mathbb{B}^2(0,1)$, the ball of unitary radius in $\mathbb{R}^2$. We are interested in exact artificial boundary conditions to be imposed at $x = -N, N$ for given $N > 0$. Since the left artificial boundary can be handled exactly in the same way as the right one, we shall only consider the latter case. Let u_N denote the solution in $[x < N]$ and u_R denote the solution in $[x > N]$. Then we can combine them into a solution of the original problem iff

$$u_N(N,y) \;=\; u_R(N,y)\,, \quad \partial_x u_N(N,y) \;=\; \partial_x u_R(N,y)\,, \quad y \in \mathbb{S}^1\,.$$

Interpreting the first condition as a boundary condition for the problem in $[x > N]$ it is not difficult to verify that its solution is given by

$$u_R(x,\cdot) \;=\; e^{-(x-N)B}u_N(N,\cdot) + \int_N^\infty B^{-1}\big(e^{-|x-\tilde{x}+N|\,B} - e^{-|x+\tilde{x}-N|\,B}\big)f(\tilde{x},\cdot)\,d\tilde{x}\,, \tag{3.17}$$

where $B = \sqrt{-\triangle_{\mathbb{B}^2}}$ and $-\triangle_{\mathbb{B}^2}$ is the abstract operator on, say, $\mathrm{L}_2\,\big(\mathbb{B}^2(0,1)\big)$ associated to the Dirichlet problem (see Subsection 1 of this section). Imposing the second glueing condition and with the help of (3.17) we therefore obtain the following exact nonlocal artificial boundary condition

$$\partial_x u_N + Bu_N \;=\; \int_N^\infty \big(e^{-|N-\tilde{x}|\,B} + e^{-|N+\tilde{x}|\,B}\big)f(\tilde{x},\cdot)\,d\tilde{x}\,.$$

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