J.evol.equ. 2 (2002) 395 – 424
1424–3199/02/040395 – 30

Journal of Evolution Equations

A 2-D free boundary problem with onset of a phase and singular elliptic boundary value problems

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This paper is dedicated to Prof. H. Amann

Abstract. Numerous models of industrial processes, such as diffusion in glassy polymers or solidification phenomena, lead to general one phase free boundary value problems with phase onset. The classical well-posedness of a fast diffusion approximation to the concerned free boundary value problems is proved. The analysis is performed via a singular change of variables leading to a singular system in a fixed domain. An existence and regularity theory for classical solutions is developed for the relevant underlying class of singular elliptic boundary value problems and is then used to prove the well-posedness for the models considered in which these are coupled to Hamilton-Jacobi or to parabolic evolution equations.

1. Introduction

In this paper we consider two dimensional generalizations of models first proposed in a 1-D setting by many different authors (cf. [**?**], [**?**], [**?**]). They arise in practical applications such as diffusion in glassy polymers, oxydation and solidification processes and other industrial problems. They are characterized by being one phase free boundary value problems, short FBPs, of diffusive type. We are especially interested in the case when the phase is initially absent but instantaneously develops as the process is started. To the best of our knowledge no rigorous analytical results concerning classical solutions are available in the literature for this class of problems in more than one space dimension. Those contained in this paper are therefore the first obtained in this direction. The main difficulties stem from a nonlinear coupling of a singular elliptic boundary value problem with a Hamilton-Jacobi equation. This structure prevents us from developing a weak solvability theory for the singular elliptic equations. This is essentially due to the loss of regularity caused by taking traces on the boundary in the case of Sobolev spaces which makes it impossible to treat the Hamilton-Jacobi in a consistent way. This is of course a consequence of the well-know lack of regularization of Hamilton-Jacobi equations.

Key words: Free boundary problems, singular elliptic boundary value problems, regularity theory, singular hölder spaces, vector-valued fourier multipliers.

The techniques developed to study the problem are based on operator-valued Fourier multipliers and spaces of singular Hölder continuous functions. A new approach to the analysis of the dilation of symbols is the core of our analysis and makes it possible to develop a regularity theory for singular elliptic boundary value problems.

When the degeneracy of the domain is not an issue, that is, when no topological change is observed, several works have been published, which utilize different techniques to establish local and/or global existence and regularity. Since we are particularly interested in classical solutions we refer to the recent papers of Escher and Simonett ([**?**], [**?**] and [**?**]) and references cited in these papers which deal with the non-degenerate case.

An asymptotic analysis of the model of interest is available, see [**?**]. In that paper a model is set up and the equations are brought into nondimensional form. Here we shall skip the nondimensionalization process. That paper also contains a formal justification for considering the quasi-stationary approximation (see below). It namely turns out to be a regular perturbation of the evolutionary system.

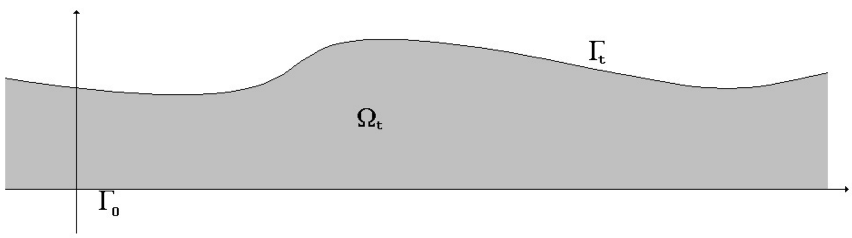
The problem consists of a diffusion equation

$$\varepsilon u_t - u_{xx} - u_{yy} = 0, \text{ in } \bigcup_{t>0} \{t\} \times \Omega_t \tag{1.1}$$

of which only the quasi-stationary approximation will be considered in this paper. The equation has to hold in an unkown strip-like domain $\Omega_t \subset \mathbb{R}^2$ with fixed boundary $\Gamma_0 = \mathbb{R} \times \{0\}$ and moving boundary

$$\Gamma_t = \{(x, s(t, x)) | x \in \mathbb{R}\}$$

for any positive time $t > 0$. Hereby we obviously assumed Γ_t to be parametrized by the unknown smooth function s. The situation is depicted in the figure below.



The degeneracy is produced by the initial condition $\Gamma_{t=0} = \Gamma_0$, or

$$s(0, \cdot) \equiv 0, \tag{1.2}$$

which implies that Ω_0 is empty. On Γ_0 we shall impose a nonhomogeneous Dirichlet condition

$$u(t, x, 0) = g(x),\ x \in \mathbb{R} \tag{1.3}$$

which models a reservoir kept at the fixed concentration $g(x)$ and which introduces two dimensional effects into the model. The problem becomes in fact 1-D if we assume g to be spacially constant. At the moving front a mass balance leads to the condition

$$-(\nabla u|\nu_t) = (1 + \varepsilon u)V,\ \text{on } \Gamma_t \tag{1.4}$$

where $\nu_t = \frac{(-s_x, 1)}{\sqrt{1+s_x^2}}$ is the unit outer normal to Γ_t and V is the front velocity observed in the normal direction. Since the concrete applications of interest present a sharp interface across which a concentration drop occurs, continuity conditions on the free boundary are inappropriate and a widely accepted phenomenological law is used instead which was introduced by Astarita and Sarti in a one dimensional setting. They assumed the front velocity to be proportional to the deviation from a threshold concentration to some power. In a higher dimensional setting curvature effects may be important and we consider a generalization of the Astarita-Sarti condition incorporating them as well. We assume that

$$V = (1 + \delta\kappa)u^n,\ \text{on } \Gamma_t \tag{1.5}$$

where n a positive integer and $\kappa = \frac{s_{xx}}{(1+s_x^2)^{3/2}}$ is the curvature of Γ_t at the point $(x, s(t, x))$. The first term is taken over from 1-D without modification and, if δ is zero, this leads to

$$s_t = \sqrt{1 + s_x^2}\, u^n,\ \text{on } \Gamma_t, \tag{1.6}$$

which is the straight forward generalization of the 1-D condition. That the second term may also be important in 2-D can be seen by geometrical considerations. If we assume that front velocity be proportional to the average concentration in a characteristic ball (of radius δ) around each point rather than to the concentration at each point we end up with the curvature correction term. This is due to the fact that the diffusing species occupies a smaller or larger portion of the ball depending on the front convexity. It enforces the intuitive observation that the front velocity should be slower at high curvature places and higher at low curvature. We consider here both negative and positive curvature and normalize it to be positive at points where the front is convex. Notice that the one dimensional behavior is locally recovered in flat regions.

The strategy we choose to attack the problem is the following. Assuming that ε is very small, we shall substitute the above equations by their quasi-stationary approximation, that is, we shall set $\varepsilon = 0$. This is a first step towards solving the evolutionary problem. The moving domain Ω_t will then be transformed to the strip $S := \mathbb{R} \times (0, 1)$ by means of the singular change of variables

$$(t, x, y) \to (\tau, \xi, \eta) := (t, x, y/s(t, x)).$$

The equations obtained will be singular. Taking the liberty of relabelling the new variables with the old names after the change of variable, equations (1.1), (1.3) and (1.4) essentially (we shall make this precise) have the following form

$$-u_{xx} - \frac{1}{t^2 g^2} u_{yy} = f, \text{ in } S, \tag{1.7}$$

$$u = g, \text{ on } \Gamma_0, \tag{1.8}$$

$$-\frac{1}{t} u_y = g^2, \text{ on } \Gamma_1, \tag{1.9}$$

where we set $\Gamma_i = \mathbb{R} \times \{i\}$, $i = 0, 1$. The whole system is in fact nonlinear and involves the unknown s, but as far as equations (1.1), (1.3) and (1.4) are concerned the system will have the specified nature, at least to "leading order". Fixing appropriate function spaces, it will turn out that the solution of (1.7)–(1.9) consists of three distinct parts with different behaviors at $t = 0$. In fact

$$u = u_P + u_N + u_D,$$

where $u_P \sim t^2$, $u_N \sim t$ and $u_D \sim t^0$. Observe, however, that

$$\partial_y^j u_N \sim t^j \text{ and } \partial_y^j u_D \sim t^j \text{ for } j = 1, 2,$$

in agreement with equations (1.7) and (1.9). The understanding of (1.7)–(1.9) will be crucial for solving the full transformed problem and to it will be dedicated much of the first part of the paper. The main techniques used to analyze (1.7)–(1.9) are those of operator valued Fourier multiplier theorems in Besov spaces combined with a novel approach to the dilation of symbols based on spaces of singular Hölder continuous functions. To deal with the remaining equation for the unknown function s we shall rely on optimal regularity results for parabolic equations if the curvature term is present and on a careful use of the method of characteristics in the event that the curvature term is absent, that is, if the equation becomes of the Hamilton-Jacobi type. Having then all the pieces together we shall produce a local classical solution to the whole system via the Banach contraction principle.

The paper is organized in two main sections. Section 2 is devoted to transforming the problem to a fixed domain problem and to developing the existence and regularity theory for anisotropic singular elliptic boundary value problems in a strip. The latter is obtained in spaces of singular and standard Hölder continuous functions. In particular the transformation to a fixed domain problem performed in Subsection 2.1 leads to a singular boundary value problem for the unknown u. The analysis of its "leading order" part in Subsection 2.2 will lead to operator valued symbols for the description of its solution. Subsection 2.3 is devoted to the analysis of those symbols on which rests the regularity theory developed in the following and last subsection of Section 2. In Section 3 we solve the full system for both the cases $\delta > 0$ and $\delta = 0$. The first subsection is devoted to proving existence for the singular boundary value problem for given front function s. In Subsection 3.2 we study the dependence of this solution on the front function. The following two subsections will be devoted one each to the case where curvature effects are not and are considered, respectively. The main results about existence and uniqueness of the solution to the quasi-stationary approximation, Theorems 3.5 and 3.7, are contained in the corresponding subsections.

2. Transformation to a fixed domain problem and the prototype singular problem

2.1. *The transformation*

Assuming that the penetration depth at time t may be described as the graph of a function $s(t,\cdot)$, which is obviously true for $t = 0$, we now reformulate the staedy-state approximation problem as a singular elliptic-parabolic/hyperbolic system for the pair (u, s) in the infinite strip $\mathbb{R} \times (0, 1)$. The singular nature of the elliptic equation stems from the chosen change of variable needed to fix the domain. We recall that

$$(\tau, \xi, \eta) := (t, x, y/s(t, x))$$

is the chosen change of variable. It becomes singular as time zero is approached and this degeneracy will be apparent in the coefficients of the transformed elliptic operator. After transforming the equations we revert to the old notation for both the independent and the dependent variables to obtain

$$-\partial_x^2 u - \frac{1}{s^2}(1 + y^2 s_x^2)\partial_y^2 u + 2y\frac{s_x}{s}\partial_x\partial_y u + y\frac{s_{xx}s - 2s_x^2}{s^2}\partial_y u = 0 \tag{2.1}$$

$$\gamma_0 u = g, \tag{2.2}$$

$$-\frac{1}{s}(1 + s_x^2)\gamma_1\partial_y u + s_x\gamma_1\partial_x u = \dot{s}, \tag{2.3}$$

$$\dot{s} = \left(\sqrt{1 + s_x^2} + \delta\frac{s_{xx}}{1 + s_x^2}\right)\gamma_1 u, \tag{2.4}$$

$$s(0, \cdot) \equiv 0, \tag{2.5}$$

where the first equation has now to be satisfied for any positive time in the strip S and γ_i denotes the restriction operator on the line Γ_i, $i = 0, 1$. We also assumed that $n = 1$ to simplify the notation. It will be clear that the general case can be treated in the same way. The singularity is now apparent in the equations, especially in the first one, and, unlike in the 1-D situation, it affects the equation in an anisotropic way. This prevents the use of the techniques developed in [**?**] for the 1-D case. The anisotropic nature of the singularity does not allow us to apply the available theory of elliptic BVPs, either. Thus the strategy to attack (2.1)–(2.5) is to first single out the "leading order" singular part of (2.1)–(2.3) and treat the nonlinear error produced as a "lower order" perturbation. As the analysis develops further it will become clear what we mean by this. First, since we are looking for a regular solution, we formally infer from equations (2.1)–(2.3) that

$$u(0, x, y) = g(x),\ (x, y) \in \mathbb{R} \times [0, 1],$$

and then from (2.4)–(2.5) that

$$\dot{s}(0, x) = g(x)$$

and finally guess that the leading asymptotic behavior of the solution is described by

$$-w_{xx} - \frac{1}{t^2 g^2} w_{yy} - \frac{g_x}{g} w_x = 0 \text{ in } S \times (0, \infty), \tag{2.6}$$

$$\gamma_0 w = g \qquad \text{on } \Gamma_0 \times (0, \infty), \tag{2.7}$$

$$-\frac{1}{t} \gamma_1 w_y = g^2 \text{ on } \Gamma_1 \times (0, \infty), \tag{2.8}$$

where the first order term in (2.6) is introduced for technical reasons which will become apparent below. The analysis of this system is our next step. To do this we have first to fix the functional setting in which we shall be working for the rest of the paper.

2.2. *The singular elliptic problem*

Let X be a Banach space. Then we denote by $\mathrm{BUC}^\alpha(\mathbb{R}, X)$ the space of bounded and uniformly Hölder continuous functions of exponent $\alpha \in (0, 1)$ endowed with its natural norm defined by

$$\|u\|_{\mathrm{BUC}^\alpha} = \|u\|_\infty + [u]_\alpha,\ u \in \mathrm{BUC}^\alpha(\mathbb{R}, X)$$

where $[u]_\alpha = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$. For $k \in \mathbb{N}$ we write $\mathrm{BUC}^{k+\alpha}(\mathbb{R}, X)$ for the space of functions for which all derivatives up the k-th order ones are in $\mathrm{BUC}^\alpha(\mathbb{R}, X)$. We assume that $g \in \mathrm{BUC}^{3+\alpha}(\mathbb{R})$ and that $g \geq g_0 > 0$ and set

$$E_0 := \mathrm{BUC}^\alpha(\mathbb{R}, \mathrm{C}([0, 1]))$$

and

$$E_1 := \{u \in \mathrm{BUC}^\alpha(\mathbb{R}, \mathrm{C}([0,1])) \big| \partial_x^j \partial_y^k u \in \mathrm{BUC}^\alpha(\mathbb{R}, \mathrm{C}([0,1])), 0 \le k + j \le 2\}$$

Further we need the change of variable G defined by

$$\tilde{x} := G^{-1}(x) = \int_0^x \frac{1}{g(\xi)}\, d\xi,$$

which transforms (2.6)–(2.8) into

$$-\widetilde{w}_{\tilde{x}\tilde{x}} - \frac{1}{t^2}\widetilde{w}_{yy} = 0 \ \text{ in } S \times (0,\infty), \tag{2.9}$$

$$\gamma_0 \widetilde{w} = g \qquad \text{on } \Gamma_0 \times (0,\infty), \tag{2.10}$$

$$-\gamma_1 \widetilde{w}_y = tg^2 \ \text{ on } \Gamma_1 \times (0,\infty) \tag{2.11}$$

for $\widetilde{w}(\tilde{x}) = w(G(\tilde{x}))$. Notice that G, in view of the assumptions on g, induces an isomorphism G^* of E_0 given by

$$G^*(w)(\tilde{x}) := w(G(\tilde{x}))$$

for which E_1 is invariant, that is

$$G^*(E_1) = E_1. \tag{2.12}$$

We denote its inverse by G_*. We are of course interested in the properties of the solution operator to system (2.6)–(2.8). We shall therefore construct the general solution to

$$-\widetilde{w}_{\tilde{x}\tilde{x}} - \frac{1}{t^2}\widetilde{w}_{yy} = f \ \text{ in } S \times (0,\infty), \tag{2.13}$$

$$\gamma_0 \widetilde{w} = g \qquad \text{on } \Gamma_0 \times (0,\infty), \tag{2.14}$$

$$-\frac{1}{t}\gamma_1 \widetilde{w}_y = h \ \text{ on } \Gamma_1 \times (0,\infty). \tag{2.15}$$

By taking a Fourier transform in the x-direction it is easily seen that the Fourier transform $\widehat{\widetilde{w}}$ of the solution is given by

$$\widehat{\widetilde{w}} = \widehat{\widetilde{w}}_P + \widehat{\widetilde{w}}_N + \widehat{\widetilde{w}}_D,$$

where

$$\widehat{\widetilde{w}}_P(t,\xi,y) = t^2((t^2\xi^2 + C)^{-1}\hat{f})(\xi, y),$$

$$\widehat{\widetilde{w}}_N(t,\xi,y) = -\frac{\sinh(t\xi y)}{\xi \cosh(t\xi)}\hat{h}$$

and

$$\widehat{\widehat{w}}_D(t,\xi,y)=\frac{\cosh(t\xi(1-y))}{\cosh(t\xi)}\hat{g}.$$

Hereby we denoted by C the realization in $F_0 := \mathrm{C}([0,1])$ of the second order operator $-\partial_y^2$ with domain of definition

$$F_1=\mathrm{C}_0^2 := \big\{ w\in \mathrm{C}^2([0,1]) \,\big|\, w(0)=0 \text{ and } w_y(1)=0\big\}.$$

It is convenient to introduce the following notation to analyze the above expressions:

$$a(\xi,y) := \frac{\cosh(\xi(1-y))}{\cosh(\xi)},\ b(\xi,y) := -\frac{\sinh(\xi y)}{\xi\cosh(\xi)},\ (\xi,y)\in\mathbb{R}\times[0,1]. \tag{2.16}$$

With

$$c(\xi) := (\xi^2+C)^{-1},\ \xi\in\mathbb{R} \tag{2.17}$$

we then have

$$\widehat{\widehat{w}}_P = t^2(t^2\xi^2+\mathrm{C})^{-1}\hat{f} = t^2(\sigma_t c)\,\hat{f}$$

and

$$\widehat{\widehat{w}}_D = (\sigma_t a)\,\hat{g} \text{ and } \widehat{\widehat{w}}_N = t(\sigma_t b)\,\hat{h},$$

where the dilation σ_t is defined by $\sigma_t f = f(t\,\cdot)$. We shall not mention explicitly the dependence of a and b on the variable y since we consider them as vector-valued symbols.

2.3. *Symbol analysis*

To analyze the above symbols we take the operator-valued Fourier multiplier point of view. In particular we shall make use of [**?**, Theorem 6.2], which is a vector-valued Fourier multiplier theorem for Besov spaces which generalizes the well known scalar Mikhlin multiplier theorem. It is important that this particular generalization does not need the UMD property in any way. Given two Banach spaces X_0 and X_1 we introduce the symbol classes

$$\mathcal{S}_k^j(X_0,X_1)= \Big\{ a\in \mathrm{C}^k(\mathbb{R},\mathcal{L}(X_0,X_1)) \,\Big|\, \sup_{\xi\in\mathbb{R}} |(1+|\xi|^2)^{(j+i)/2} a^{(i)}(\xi)|_{\mathcal{L}(X_0,X_1)} < \infty,\ i\le k\Big\}, \tag{2.18}$$

for $j\in\mathbb{Z}$ and $k\in\mathbb{N}\cup\{\infty\}$. Denoting by $\mathrm{B}^s_{p,q}(\mathbb{R},X_j)$, for $s\in\mathbb{R}$, $p,q\in[1,\infty]$ and $j=0,1$, the vector valued Besov space defined in [**?**] we can specialize the main theorem of that paper.

THEOREM 2.1. *Assume that* $a \in \mathcal{S}_2^j(X_0, X_1)$ *for some* $j \in \mathbb{Z}$. *Then*

$$\mathcal{F}^{-1} a \mathcal{F} \in \mathcal{L}(\mathrm{B}^s_{p,q}(\mathbb{R}, X_0), \mathrm{B}^{s+j}_{p,q}(\mathbb{R}, X_1))$$

for $s \in \mathbb{R}$ *and* $p, q \in [1, \infty]$.

In particular if $a \in \mathcal{S}_2^j(X_0, X_1)$ for some $j \in \mathbb{Z}$ we obtain

$$\mathcal{F}^{-1} a \mathcal{F} \in \mathcal{L}(\mathrm{BUC}^{n+\alpha}(\mathbb{R}, X_0), \mathrm{BUC}^{n+j+\alpha}(\mathbb{R}, X_1)),$$

for any $n \in \mathbb{N}$ and $\alpha \in (0, 1)$ such that $n + j \geq 0$ by choosing $s = n + \alpha$ and $p = q = \infty$ as follows from [**?**, Section 5]. Here we only used that spaces of Hölder continuous functions are special instances of Besov spaces. We shall need the following remark.

REMARK 2.1. The operator C is an invertible sectorial operator on $\mathrm{C}([0, 1])$ with domain of definition F_1.

Proof. A proof of this result can be found in [**?**], Corollaries 3.1.21 and 3.1.24. We remark here that this is valid only in one space dimension. □

A careful analysis of the symbols a, b and c shows that

LEMMA 2.2. *Let the symbols a, b and c be given as in* (2.16) *and in* (2.17). *Then*

$$a \in \mathcal{S}_\infty^{-k}(\mathbb{R}, \mathrm{C}^k([0, 1])),\ b \in \mathcal{S}_\infty^{-k+1}(\mathbb{R}, \mathrm{C}^k([0, 1]))$$
$$\text{and } c \in \mathcal{S}_\infty^{-k+2}(\mathrm{C}([0, 1]), \mathrm{C}^k([0, 1]))$$

for $k = 0, 1, 2$.

Proof. We first remark that we only need estimates for the symbols with the weight ξ to some power instead of the weight $(1 + |\xi|^2)^{1/2}$ to the appropriate power used in the definition of the symbol classes. This is possible because the symbols under consideration are not singular in the origin. Since C is an invertible, sectorial operator on F_0 with domain of definition F_1, we have that $\|(\lambda + \mathrm{C})^{-1}\|_{\mathcal{L}(F_0, F_j)} \leq c \frac{1}{(1+|\lambda|)^{1-j}}$ for λ in some sector of the complex plain containing the positive real half line and $j = 0, 1$. Using interpolation inequalities it is easy to conclude that

$$\|\partial_y^k \xi^{2-k} (\xi^2 + \mathrm{C})^{-1}\|_{\mathcal{L}(F_0)} < \infty,\ \xi \in \mathbb{R}$$

for $k = 0, 1, 2$. Since $\xi^i (\partial_\xi^i c)(\xi) = \sum_{1 \leq l \leq i} p_l(\xi) c^{l+1}$ for some polynomials p_l of order at most $2l$ we conclude that

$$\sup_{\xi \in \mathbb{R}} \|\xi^{i+k} \partial_y^{2-k} (\partial_\xi^i c)(\xi)\|_{\mathcal{L}(F_0)} < \infty$$

which concludes the proof for the symbol c. Let us now consider the symbol a. It is easily seen that

$$\sup_{\xi\in\mathbb{R}}\sup_{y\in[0,1]}|a(\xi,y)|<\infty.$$

As for the absolute value $|(\partial_\xi a)(\xi,y)|$ of the first derivative

$$\partial_\xi a(\xi,y)=\frac{1}{\cosh(\xi)}[(1-y)\sinh((1-y)\xi)-\tanh(\xi)\cosh((1-y)\xi)]$$

it attains its maximum, as a function of y, where

$$-\sinh((1-y)\xi)-\xi(1-y)\cosh((1-y)\xi)+\xi\sinh((1-y)\xi)\tanh(\xi)=0,$$

or, equivalently, where

$$1=\frac{\tanh((1-y)\xi)}{(1-y)\xi}[\xi\tanh(\xi)-1]. \tag{2.19}$$

The latter implies that there exists a $\xi_0>0$, independent of y, such that the maximum is attained at $y=1$ if $|\xi|\le\xi_0$. In fact the first term of the product on the right hand side is always bounded by 1 whereas the second is only larger than 1 if ξ is chosen large enough. In other words, the maximum is reached at an interior point only if ξ is large enough. In that case, that is, when $|\xi|>\xi_0$, the maximum moves to the left and, asymptotically, $y_{\max}\approx\frac{1}{\xi}$. This is seen as follows. First observe that the function is symmetric in the Fourier variable which allows us to consider the case $\xi>0$ only. For large ξ we have

$$\tanh(\xi)=1+O(e^{-2\xi})$$

and (2.19) can be rewritten as

$$(1-y)\xi=\xi-1+O(\xi e^{-2\xi})+O(e^{-2(1-y)\xi}).$$

This gives $y\approx\frac{1}{\xi}$, which, for large ξ, is consistent with the asymptotic expansion. In particular the last term cannot be of order 1. Thus we have

$$\sup_{\xi\in\mathbb{R}}\|\xi(\partial_\xi a)(\xi,\cdot)\|_\infty\le\max\{\sup_{|\xi|\le\xi_0}\|\xi(\partial_\xi a)(\xi,\cdot)\|_\infty,\ \sup_{|\xi|>\xi_0}\|\xi(\partial_\xi a)(\xi,\cdot)\|_\infty\},$$

where the first can easily be bounded, whereas for the second we have

$$\xi(\partial_\xi a)(\xi,y_{\max})\approx\frac{\cosh(\xi-1)}{\cosh(\xi)}[(\xi-1)\tanh(\xi-1)-\xi\tanh(\xi)]$$

which is bounded in absolute value (see below). Now, since

$$\begin{aligned}\partial_\xi a(\xi,y)&=\frac{\cosh((1-y)\xi)}{\cosh(\xi)}[(1-y)\tanh((1-y)\xi)-\tanh(\xi)]\\&=a(\xi,y)[(1-y)\tanh((1-y)\xi)-\tanh(\xi)]=a(\xi,y)d(\xi,y)\end{aligned}$$

for d defined in an obvious way, we obtain

$$(\partial_\xi^2 a)(\xi, y) = a(\xi, y)[d^2(\xi, y) + (\partial_\xi d)(\xi, y)]$$
$$= \frac{1}{\cosh(\xi)}[\cosh((1-y)\xi)((1-y)^2 - 1 + 2\tanh^2(\xi))$$
$$- 2(1-y)\sinh((1-y)\xi)\tanh(\xi)]$$

Now, as before, a continuity argument gives us a bound for ξ in any given compact subset of $\mathbb{R}^n$. For large ξ we compute the first y derivative of the second term in the second product to obtain the necessary condition

$$-\xi\tanh((1-y)\ (\xi))((1-y)^2 - 1 + 2\tanh^2(\xi))$$
$$+ 2\tanh((1-y)\ (\xi))\tanh(\xi) - 2(1-y) + 2\xi(1-y)\tanh(\xi) = 0$$

By the same asymptotic argument used above we see that the last equation can asymptotically be replaced by

$$-\xi((1-y)^2 + 1) + 2 - 2(1-y) + 2\xi(1-y) = 0$$

which gives

$$y_{\max} \approx 0 \text{ or } y_{\max} \approx 2/\xi.$$

Only the second case is of interest since the symbol of interest is bounded on the y-boundary. Substituting the second expression for the maximum in the symbol we obtain

$$\xi^2(\partial_\xi^2 a)(\xi, y_{\max}) \approx \frac{\cosh(\xi-2)}{\cosh(\xi)}[(\xi-2)\tanh(\xi-2) - \xi\tanh(\xi)]^2$$
$$+ \frac{\cosh(\xi-2)}{\cosh(\xi)}[(\xi-2)^2(1-\tanh(\xi-2)) - \xi^2(1-\tanh(\xi))].$$

The two terms on the right hand side can now be bounded separately. Observe that the second term decays as ξ grows whereas the first is only bounded. The claim for $k = 0$ concerning a would now follows by similar computations for the higher derivatives. It is however easier to proceed as follows. Recall that

$$\xi\partial_\xi a = a\xi d \text{ and } \xi^2\partial_\xi^2 a = a(\xi d)^2 + a\xi^2\partial_\xi d$$

and observe that the second term on the right hand side of the second equality is easily bounded in view of

$$\xi^2\partial_\xi d = \xi^2(1-y)^2(1-\tanh^2((1-y)\xi) - \xi^2(1-\tanh^2(\xi))$$

and the decay properties of the function $1 - \tanh^2$. Writing a in terms of exponential functions and observing that

$$\xi d \approx -\xi y$$

it is easily seen that $a(\xi d)$ and $a(\xi d)^2$ are bounded, too. Now it can be seen by induction that a similar structure is shared by the higher order derivatives, that is,

$$\xi^n \partial_\xi^{(n)} = a(\xi d)^n + \xi^n(\dots)$$

where the dots stand for terms containing derivatives of d which have decay properties similar to those of $1 - \tanh^2$. The same observation as above shows that the first term is also bounded. This concludes the proof in the case $k = 0$ for the symbol a.

It is obvious that $[\xi \mapsto \xi b]$ is bounded in the supremum norm. The rest follows from similar computations as above, from $(\partial_y b)(\xi, y) = a(\xi, 1-y)$ and $(\partial_y^2 b)(\xi, y) = \xi^2 b(\xi, y)$ and from similar relations for $\partial_y^j a$, $j = 1, 2$. □

REMARK 2.2. It is an interesting observation that the previous lemma gives, using theorem 6.2 in [**?**] a novel regularity result concerning the strip problem for the Laplacian. In fact we have

$$(-\triangle, \gamma_0, \gamma_1 \partial_y) \in \mathcal{L}_{is}(E_1, E_0 \times \mathrm{BUC}^{2+\alpha}(\mathbb{R}) \times \mathrm{BUC}^{1+\alpha}(\mathbb{R})).$$

It is standard to extend this result to more general elliptic boundary value problems in the strip.

Define now $H_1 := \mathcal{F}^{-1}a$ and $H_2 := \mathcal{F}^{-1}b$. Then, from $a(0, \cdot) \equiv 1$ and from $b(0, \cdot) = \mathrm{id}_{[0,1]}$ it follows that

$$\frac{1}{t}\sigma_t H_1 \longrightarrow \delta, \ \frac{1}{t}\sigma_t H_2 \longrightarrow -\delta \,\mathrm{id}_{[0,1]} \text{ in } \mathcal{S}'(\mathbb{R}, \mathrm{C}([0, 1]))$$

as t tends to zero. By $\mathcal{S}'(\mathbb{R}, \mathrm{C}([0, 1]))$ we denoted the space of vector valued tempered distributions endowed with its natural locally convex topology. The convergence to the given limits follow from the pointwise convergence of the symbols and Lebesgue theorem together with the fact that the vector valued Fourier transform is an isomorphism, i.e.,

$$\mathcal{F} \in \mathcal{L}_{is}(\mathcal{S}'(\mathbb{R}, \mathrm{C}([0, 1])), \mathcal{S}'(\mathbb{R}, \mathrm{C}([0, 1]))).$$

A proof of this fact can be found in [**?**] or in the original work [**?**] by Schwartz. We are now in a position to prove the following

PROPOSITION 2.3. *Assume that*

$$(f, g, h) \in \mathrm{BUC}^\alpha(\mathbb{R}, F_0) \times \mathrm{BUC}^{2+\alpha}(\mathbb{R}) \times \mathrm{BUC}^{1+\alpha}(\mathbb{R})$$

and let $\widetilde{w}$ *be the solution of* (2.13)–(2.15). *Then one has*

$$\widetilde{w} = g - t\, \mathrm{id}_{[0,1]} h + O(t^2) \text{ in } E_0 \text{ as } t \to 0.$$

Proof. We shall first prove that

$$\widetilde{w}_D = g + O(t^2) \text{ and } \widetilde{w}_N = y\,t\,g^2 + O(t^2) \text{ in } E_0.$$

and then that $\widetilde{w}_P = O(t^2)$ as $t \to 0$. Since $a(0,\cdot) \equiv 1$ and $(\partial_\xi a)(0,\cdot) \equiv 0$ we can write

$$\sigma_t a - 1 = \int_0^t \int_0^\tau \xi^2 \sigma_\eta (\partial_\xi^2 a)\, d\eta\, d\tau.$$

The fact that $a \in \mathcal{S}^0(\mathbb{R}, F_0)$ implies that

$$[\eta \mapsto \mathcal{F}^{-1}(\xi^2 \sigma_\eta (\partial_\xi^2 a))\mathcal{F}] \in \mathrm{B}(\mathbb{R}, \mathcal{L}(\mathrm{BUC}^{2+\alpha}(\mathbb{R}), \mathrm{BUC}^{\alpha}(\mathbb{R}, F_0))),$$

from which we conclude that

$$\widetilde{w}_D - g = O(t^2) \text{ in } E_0 \text{ as } t \to 0$$

since $g \in \mathrm{BUC}^{2+\alpha}(\mathbb{R})$. As for $\widetilde{w}_N$, arguing in a similar manner, the assertion follows from

$$t\sigma_t b - t\,\mathrm{id}_{[0,1]} = t \int_0^t \xi \sigma_\tau \partial_\xi b\, d\tau,$$

from $b \in \mathcal{S}^1(\mathbb{R}, F_0)$ and from $h \in \mathrm{BUC}^{1+\alpha}(\mathbb{R})$. Finally the estimate

$$\widetilde{w}_P = O(t^2)$$

easily follows from the properties of the symbol c and from $\hat{\widetilde{w}}_P = t^2 \sigma_t c \hat{f}$. □

In particular, observing that $u = G_* \widetilde{u}$, we conclude that

COROLLARY 2.4. *Let u be the solution of* (2.6)–(2.8). *Then*

$$u = g - tyg^2 + O(t^2) \text{ in } E_0.$$

2.4. *Regularity theory*

The last proposition of the previous section still does not give us a full picture about the behavior of the solution operator to (2.6)–(2.8). To complete the analysis we need to understand what is the role played by the time variable. In particular, we want to consider time dependent data f, g and h. Making use of theorem 6.2 in [**?**] it is easy to obtain, for fixed $t > 0$, that

$$A(t) := G^* \mathcal{F}^{-1} \left(\xi^2 + \frac{\mathrm{C}}{t^2} \right) \mathcal{F} G_* \in \mathcal{L}(E_{1,b}, E_0) \tag{2.20}$$

with $\sup_{t\in[0,1]} \|A(t)^{-1}\|_{\mathcal{L}(E_0,E_{1,b})} \le c$, where

$$E_{1,b} = \big\{u \in E_1 \,\big|\, u(\cdot,0) = 0 \text{ and } \partial_y u(\cdot,1) = 0\big\}.$$

However, a deeper understanding of the mapping properties of the solution operator is essential. We shall make extensive use of the following notation:

$$R_D := G_* \mathcal{F}^{-1}(\sigma_t a \cdot)\mathcal{F}G^* \text{ and } R_N := G_* \mathcal{F}^{-1}(\sigma_t t b \cdot)\mathcal{F}G^*, \tag{2.21}$$

so that the solution operator R associated to the BVP (2.6)–(2.8) can be written as

$$R(f,g,h) = A^{-1}f + R_D g + R_N h.$$

Some of its properties are collected in the next few lemmas and propositions. If we set

$$\|a\|_{\mathcal{S}_2^j} = \max_{i\le 2} \sup_{\xi\in\mathbb{R}} |(1+|\xi|^2)^{(j+i)/2} a^{(i)}(\xi)|_{\mathcal{L}(X_0,X_1)}$$

then $\mathcal{S}_2^j(X_0,X_1)$ becomes a Banach space. Now, given a Banach space X we denote by

$$\mathrm{C}_1^{1-}(J,X)$$

the space of bounded functions φ for which $[t \mapsto t\varphi(t)]$ is Lipschitz continuous on a compact interval J. It becomes a Banach space if endowed with the norm given by

$$\|\varphi\|_{\mathrm{C}_1^{1-}} = \|\varphi\|_\infty + [(\cdot)\varphi(\cdot)]_1$$

where $[(\cdot)\varphi(\cdot)]_1 = \sup_{t\ne s} \frac{\|t\varphi(t)-s\varphi(s)\|}{|t-s|}$.

LEMMA 2.5. *Assume that* $a \in \mathcal{S}_2^j(X_0,X_1)$ *for some* $j \in \mathbb{N}$. *Then*

$$[t \mapsto t^j \sigma_t a] \in \mathrm{C}_1^{1-}(J, \mathcal{S}_2^j(X_0,X_1)),$$

which, in its turn, implies that

$$\begin{aligned} &t^j G_* \mathcal{F}^{-1}\sigma_t a \mathcal{F}G^* = t^{j-1}\sigma_{1/t} G_* \mathcal{F}^{-1} a \mathcal{F}G^* \\ &\quad \in \mathrm{C}_1^{1-}(J, \mathcal{L}(\mathrm{BUC}^\alpha(\mathbb{R},X_0), \mathrm{BUC}^{j+\alpha}(\mathbb{R},X_1))) \end{aligned}$$

Proof. Recall that the assumption implies

$$\sup_{\xi\in\mathbb{R}} \|(1+|\xi|^2)^{(j+i)/2} a^{(i)}(\xi)\|_{\mathcal{L}(X_0,X_1)} \le c < \infty$$

for $i \le 2$. Thus for $\sigma_t a$ one has

$$\partial_\xi^i (t^j \sigma_t a) = t^{j+i}\sigma_t a^{(i)} \text{ and } \partial_t(t^j\sigma_t a) = j t^{j-1}\sigma_t a + t^j \xi \sigma_t a',$$

which together with the assumption implies

$$\sup_{\xi\in\mathbb{R}} \|(1+|\xi|^2)^{(j+i)/2}\partial_\xi^i(t^j\sigma_t a)\|_{\mathcal{L}(X_0,X_1)} \le c < \infty$$

and

$$\sup_{\xi\in\mathbb{R}} t\,\|(1+|\xi|^2)^{(j+i)/2}\partial_\xi^i\partial_t(t^j\sigma_t a)\|_{\mathcal{L}(X_0,X_1)} \le c < \infty.$$

It is important that the constant c is independent of t. The first assertion now follows easily from

$$\sigma_t a - \sigma_s a = \int_s^t \partial_t(\sigma_t a)(\tau)\,d\tau$$

and the above inequalities. The second assertion is a consequence of the properties of G^*, see (2.12). □

Next we need to introduce spaces of singular and non singular Hölder continuous functions. Let $J = [0, T]$ for $T > 0$ and X be a Banach space. For $\alpha \in (0, 1)$ and $\beta \in \mathbb{R}$ we define

$$\mathrm{C}^\alpha_\beta(J,X) = \big\{u \in \mathrm{B}(J,X)\,\big|\,[t \mapsto t^\beta u(t)] \in \mathrm{C}^\alpha(J,X)\big\}$$
$$\mathrm{C}^\alpha_0(J,X) = \big\{u \in \mathrm{C}^\alpha(J,X)\,\big|\,u(0) = 0\big\}$$
$$\mathrm{C}^\alpha_{\beta,0}(J,X) = \big\{u \in \mathrm{B}(J,X)\,\big|\,[t \mapsto t^\beta u(t)] \in \mathrm{C}^\alpha_0(J,X)\big\}$$

Hereby we denoted by $\mathrm{B}(J, X)$ the space of bounded functions and by $\mathrm{C}^\alpha(J, X)$ the space of Hölder continuous functions of exponent α. We endow the singular spaces with the norm

$$\|u\|_{\mathrm{C}^\alpha_\beta} = \|u\|_\infty + [(\cdot)^\beta u(\cdot)]_\alpha,$$

where $[(\cdot)^\beta v(\cdot)]_\alpha = \sup_{t\neq s} \frac{\|t^\beta v(t)-s^\beta v(s)\|_X}{|t-s|^\alpha}$. In the sequel we shall occasionally make use of the shortened notation

$$\mathrm{C}^\alpha_\beta = \mathrm{C}^\alpha_\beta(J,X).$$

The next lemma highlights almost trivial properties of the singular continuous spaces which will be very convenient to refer to in many future proofs.

LEMMA 2.6. *The following multiplication maps are continuous*

$$\mathrm{C}^\alpha_\alpha \times \mathrm{C}^\beta_\beta \longrightarrow \mathrm{C}^\beta_\beta,\ (u,v) \mapsto uv$$
$$\mathrm{C}^\alpha_\alpha \times \mathrm{C}^\beta_0 \longrightarrow \mathrm{C}^\beta_0,\ (u,v) \mapsto uv$$

for any given $\alpha, \beta \in (0,1) \cup \{1-\}$ *with* $\alpha \geq \beta$. *For the same choice of* α *and* β *the multiplications*

$$\mathrm{C}_0^\alpha \times \mathrm{C}_0^\beta \longrightarrow \mathrm{C}_{-\alpha}^\beta,\ (u,v) \mapsto uv$$
$$\mathrm{C}_0^\alpha \times \mathrm{C}_\beta^\beta \longrightarrow \mathrm{C}_{\beta-\alpha}^\beta,\ (u,v) \mapsto uv$$

are also continuous.

Proof. We first observe that $\mathrm{C}_\alpha^\alpha \hookrightarrow \mathrm{C}_\beta^\beta$ for $\beta \leq \alpha \in (0,1) \cup \{1-\}$. Thus it suffices to prove the first continuity assertion for the case $\alpha = \beta$. Let $u, v \in \mathrm{C}_\beta^\beta$, then

$$\begin{aligned}
&|t^\beta u(t)v(t) - s^\beta u(s)v(s)| \leq |u(t)(t^\beta v(t) - s^\beta v(s))| + |(u(t) - u(s))s^\beta v(s)| \\
&\quad \leq \|u\|_\infty [t^\beta v]_\beta (t-s)^\beta + \|u\|_{\mathrm{C}_\beta^\beta} \|v\|_\infty \frac{s^\beta (t-s)^\beta}{t^\beta} \\
&\quad \leq \|u\|_{\mathrm{C}_\beta^\beta} \|v\|_{\mathrm{C}_\beta^\beta} (t-s)^\beta,
\end{aligned}$$

where the second to the last inequality follows from

$$\|u(t) - u(s)\| \leq \|u\|_{\mathrm{C}_\beta^\beta} \frac{(t-s)^\beta}{t^\beta},$$

which is in its turn a direct consequence of the definition of the norm in C_β^β. Now the desired continuity follows from the trivial inequality

$$\|uv\|_\infty \leq \|u\|_\infty \|v\|_\infty.$$

As to the second multiplication it suffices to consider the regularity of the product at zero, since away from zero the product reduces to the product of Hölder continuous functions. There one has

$$|u(t)v(t)| \leq \|u\|_\infty [v]_\beta t^\beta,$$

which readily entails the assertion. Since the last two multiplications can be handled in a similar way, we just prove the last one. Let $u \in \mathrm{C}_0^\alpha$ and $v \in \mathrm{C}_\beta^\beta$, then it is obvious that $\|uv\|_\infty \leq \|u\|_\infty \|v\|_\infty$. Furthermore we have

$$\begin{aligned}
&|t^{\beta-\alpha} u(t)v(t) - s^{\beta-\alpha} u(s)v(s)| \\
&\quad \leq |(t^{-\alpha} - s^{-\alpha}) t^\beta v(t)| + |(t^\beta v(t) - s^\beta v(s)) s^{-\alpha} u(s)| \\
&\quad \leq 2[u]_\alpha \|v\|_\infty t^{\beta-\alpha} (t-s)^\alpha + [u]_\alpha [(\cdot)^\beta v(\cdot)]_\beta (t-s)^\beta
\end{aligned} \tag{2.22}$$

and thus the assertion follows. ☐

LEMMA 2.7. *Assume that A is given by* (2.20) *and set* $J = [0, 1]$. *Then*

$$\partial_x^j \left(\frac{\partial_y}{t}\right)^k A^{-1} \in \mathrm{C}_1^{1-}(J, \mathcal{L}(E_0)),$$

for nonnegative j *and* k *with* $j + k \leq 2$.

Proof. As the Fourier symbol of $G^* A^{-1} G_*$ is given by $t^2 \sigma_t c$ and

$$c \in \mathcal{S}^{2-k}(\mathrm{C}([0, 1]), \mathrm{C}^k([0, 1])) \text{ for } k = 0, 1, 2$$

the claim follows from Lemmas 2.2 and 2.5 as well as from the mapping properties of G^*. □

COROLLARY 2.8. *Define* $E_{10} = \mathrm{BUC}^{2+\alpha}(\mathbb{R}, F_0)$ *and* $E_{01} = \mathrm{BUC}^{\alpha}(\mathbb{R}, F_1)$. *Then*

$$A^{-1} \in \mathrm{C}_1^{1-}(J, \mathcal{L}(E_0, E_{10})) \cap \mathrm{C}_{-1}^{1-}(J, \mathcal{L}(E_0, E_{01})).$$

Moreover

$$A^{-1} \in \mathrm{C}_0^{1}(J, \mathcal{L}(\mathrm{BUC}^{k+\alpha}(\mathbb{R}, F_0), \mathrm{BUC}^{k+1+\alpha}(\mathbb{R}, F_0)))$$

for $k = 0, 1$.

PROPOSITION 2.9. *Assume that the same assumptions as in the previous lemma are satisfied and let* $\beta \in (0, 1)$. *Then*

$$\partial_x^j \left(\frac{\partial_y}{t}\right)^k A^{-1} \in \mathcal{L}(\mathrm{C}_\beta^\beta(J, E_0)) \cap \mathcal{L}(\mathrm{C}_0^\beta(J, E_0))$$

for nonnegative j *and* k *with* $j + k \leq 2$.

Proof. The assertions are now an easy consequence of Lemmas 2.2, 2.5 and 2.6. □

COROLLARY 2.10. *Defining* E_{10} *and* E_{01} *as in Corollary 2.8. we obtain*

$$A^{-1} \in \mathcal{L}(\mathrm{C}_\beta^\beta(J, E_0),\ \mathrm{C}_\beta^\beta(J, E_{10}) \cap \mathrm{C}_{\beta-2}^\beta(J, E_{01})) \tag{2.23}$$

$$A^{-1} \in \mathcal{L}(\mathrm{C}_0^\beta(J, E_0),\ \mathrm{C}_0^\beta(J, E_{10}) \cap \mathrm{C}_{0,-2}^\beta(J, E_{01})) \tag{2.24}$$

The following remark will be crucial in proving existence of a solution to problem (2.1)–(2.5) in the next section.

REMARK 2.3. If $\mathrm{C}_0^\beta(J, E_0)$ is replaced by $\mathrm{C}_r^\beta(J, E_0)$, that is, the space of functions f which are Hölder continuous of exponent β with respect to the topology of E_0 and for which $f(0) \in E_{10}$, then one has

$$A^{-1} \in \mathcal{L}(\mathrm{C}_r^\beta(J, E_0), \mathrm{C}_0^\beta(J, E_{10})).$$

Proof. In view of the properties of the symbol c it is a consequence of the previous lemmas that

$$A^{-1} \in \mathcal{L}(\mathrm{C}^{\beta}(J, E_{10}), \mathrm{C}_0^{\beta}(J, E_{10})).$$

Thus

$$A^{-1}(t) f(t) = A^{-1}(f(t) - f(0)) + A^{-1} f(0)$$

entails the assertion since $f(\cdot) - f(0) \in \mathrm{C}_0^{\beta}(J, E_0)$. □

To simplify the notation we shall make use of the following abbreviation:

$$\mathrm{C}^{?}_{?,?}\,\mathrm{BUC}^{?}\,\mathrm{C}^{?}_{?} := \mathrm{C}^{?}_{?,?}(J, \mathrm{BUC}^{?}(\mathbb{R}, \mathrm{C}^{?}_{?}([0,1]))),$$

where the question marks might be replaced by any superscript or subscript. Let now $\alpha, \beta \in (0,1)$ and define

$$\mathbb{E}_0 = \mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{\alpha}\,\mathrm{C} \times \mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{2+\alpha} \times \mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{1+\alpha} \tag{2.25}$$

and

$$\mathbb{E}_1 = \Big\{ u \in \mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{2+\alpha}\,\mathrm{C} \cap \mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{\alpha}\,\mathrm{C}^2 \;\Big|\; \partial_x^j \left(\frac{\partial_y}{t}\right)^k u \in \mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{\alpha}\,\mathrm{C},\ 0 \le j+k \le 2 \Big\}. \tag{2.26}$$

We furthermore denote by $\mathring{\mathbb{E}}_0$ and $\mathring{\mathbb{E}}_1$ the spaces obtained from the above replacing the subscript β by 0. The next theorem deals with the "boundary terms".

PROPOSITION 2.11. *Let R_D and R_N be defined as in* (2.21). *Then*

$$R_D \in \mathcal{L}(\mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{2+\alpha}, \mathbb{E}_1) \cap \mathcal{L}(\mathrm{C}^{\beta}_{0}\,\mathrm{BUC}^{2+\alpha}, \mathring{\mathbb{E}}_1) \tag{2.27}$$

$$R_N \in \mathcal{L}(\mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{1+\alpha}, \mathbb{E}_1) \cap \mathcal{L}(\mathrm{C}^{\beta}_{0}\,\mathrm{BUC}^{1+\alpha}, \mathring{\mathbb{E}}_1) \tag{2.28}$$

Moreover, for $k \ge 2$,

$$R_D \in \mathcal{L}(\mathrm{BUC}^{k+\alpha}, \mathrm{C}^{1-}\,\mathrm{BUC}^{k-1+\alpha}\,\mathrm{C}) \tag{2.29}$$

$$R_N \in \mathcal{L}(\mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{k-1+\alpha}, \mathrm{C}^{\beta}_{\beta-1}\,\mathrm{BUC}^{k-1+\alpha}\,\mathrm{C}) \tag{2.30}$$

$$R_N \in \mathcal{L}(\mathrm{C}^{\beta}_{0}\,\mathrm{BUC}^{k-1+\alpha}, \mathrm{C}^{\beta}_{-1}\,\mathrm{BUC}^{k-1+\alpha}\,\mathrm{C}) \tag{2.31}$$

Proof. Since the symbols of $G_* R_J G^*$ $(J = D, N)$ are given by $\sigma_t a$ and $t\sigma_t b$ for a and b as in (2.16) the inclusions follow from Lemmas 2.2, 2.5 and 2.6, from $\partial_y^l a \in \mathcal{S}_2^{-l}(\mathbb{R}, \mathrm{C})$ and $\partial_y^l b \in \mathcal{S}_2^{-l-1}(\mathbb{R}, \mathrm{C})$ and from the properties of G^*. □

COROLLARY 2.12. *Let* $\beta \in (0, 1)$. *Then*

$$u_D \in \mathrm{C}^\beta(J, E_1) \text{ and } u_N \in \mathrm{C}^\beta_{\beta-1}(J, E_1) \tag{2.32}$$

The next theorem recapitulates the mapping properties of R.

THEOREM 2.13. *Let R be the solution operator corresponding to the boundary value problem for* $(A, \gamma_0, \frac{1}{(\cdot)}\gamma_1 \partial_y)$. *Then*

$$\begin{aligned} R \in\ & \mathcal{L}_{\mathrm{is}}(\mathbb{E}_0 \times \mathrm{C}^\beta_\beta \mathrm{BUC}^{2+\alpha} \times \mathrm{C}^\beta_\beta \mathrm{BUC}^{1+\alpha}, \mathbb{E}_1) \\ & \cap \mathcal{L}_{\mathrm{is}}(\mathring{\mathbb{E}}_0 \times \mathrm{C}^\beta_0 \mathrm{BUC}^{2+\alpha} \times \mathrm{C}^\beta_0 \mathrm{BUC}^{1+\alpha}, \mathring{\mathbb{E}}_1). \end{aligned}$$

Proof. The assertion is a consequence of Propositions 2.9 and 2.11. □

3. Existence of a local solution

This section is organized in the following way: Firstly, the singular boundary value problem (2.1)–(2.3) is considered for a fixed given front function $s \in \mathbb{B}_{\mathbb{S}}(gt, g_0/2)$ where

$$\mathbb{S} = \left\{ s \in \mathrm{C}^{1/2}(J, \mathrm{BUC}^{k(\delta)+\alpha}) \cap \mathrm{C}^{1+1/2}(J, \mathrm{BUC}^{1+\alpha}) \,\middle|\, s(0) = 0 \text{ and } \dot{s}(0) = g \right\}$$

and $\mathbb{B}(c, r)$ stands for the ball centered at c with radius r. Recall that the assumptions on the datum are

$$g \in \mathrm{BUC}^{3+\alpha}(\mathbb{R}) \text{ with } g \geq g_0 > 0.$$

The parameter k is assumed be equal to 2 if $\delta = 0$ and equal to 3, otherwise. The existence of a solution is proved and its dependence on the function s is discussed. Secondly the Hamilton-Jacobi equation ($\delta = 0$) and the parabolic evolution equation (otherwise) (2.4)–(2.5) are analyzed for a fixed given $u \in \mathbb{E}_1 \cap \mathrm{C}^{1/2}(J, \mathrm{BUC}^{2+\alpha}\,\mathrm{C})$. Again existence of a solution is obtained and its dependence on u is studied. Finally the full system is attacked and a solution is produced via the Banach contraction principle.

3.1. *Existence for the singular boundary value problem*

Let us start by considering (2.1)–(2.3). A solution will be sought in the form $u = w + u_D + u_N =: w + v$ where, we recall,

$$u_D = G_* \mathcal{F}^{-1}(\sigma_t a) * G^* g$$

and

$$u_N = G_* \mathcal{F}^{-1}(t\sigma_t b) * G^* g^2$$

This leads to the following system for w:

$$-w_{xx} - \frac{1}{t^2 g^2} w_{yy} - \frac{g_x}{g} w_x = F(w, s) \qquad \text{in } S \times J \tag{3.1}$$

$$w = 0 \text{ on } \Gamma_0 \times J \tag{3.2}$$

$$-\frac{1}{t} w_y = G(w, s, \dot{s}) \text{ on } \Gamma_1 \times J \tag{3.3}$$

where

$$F(w, s) := \left(\frac{1 + y^2 s_x^2}{s^2} - \frac{1}{t^2 g^2} \right) (w + v)_{yy} - \frac{g_x}{g} (w + v)_x$$
$$+ 2y \frac{s_x}{s} (w + v)_{xy} + y^2 \frac{s_{xx} s - 2 s_x^2}{s^2} (w + v)_y$$

and

$$G(w, s, \dot{s}) = \frac{1}{t} (s_x^2 \gamma_1 (w + v)_y - s s_x \gamma_1 (w + v)_x + s \dot{s} - t g^2).$$

Now it is easily seen that w is the fixed point of

$$w = A^{-1} F(w, s) + R_N G(w, s, \dot{s}),$$

Let $\alpha \in (0, 1)$. The next theorem shows that a solution can be found in the space $\mathbb{E}_1$.

THEOREM 3.1. *Given $s \in \mathbb{S}$ there exists a unique solution u_s of* (2.1)–(2.3) *with $u \in \mathbb{E}_1$. Moreover $u_s \in \mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}\,\mathrm{C}$ with*

$$\|u\|_{\mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}\,\mathrm{C}} \le c\, \|u\|_{\mathbb{E}_1}$$

In particular,

$$\gamma_1 u_s \in \mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha} \;\; \textit{with } \gamma_1 u_s(0) = g.$$

Proof. It suffices to construct w since then $u = u_D + u_N + w$ is the solution with the required properties. Notice that since both F and G are affine maps for fixed s it is trivial to prove that $\Phi = A^{-1} F + R_N G$ is a self map on $\mathbb{E}_{1,0} = \{w \in \mathbb{E}_1 \mid \gamma_0 w = 0\}$. It is indeed a straightforward application of the regularity results obtained in the previous section. We shall, however, need more in the sequel and we shall therefore proceed in another way as to prove that Φ maps any ball $\mathbb{B}_{\mathbb{E}_{1,0}}(0, r)$, $r >> 0$, into itself as long as the underlying time interval $J = [0, T]$ is taken small enough and r large enough. Contractivity of Φ will also be proved on a small time assumption. This will imply the existence of a unique solution in the given class. Finally we shall show that the solution satisfies the additional regularity property. Since, as far as regularity is concerned, the isomorphism G^* does not play any role, we can assume that $G^* = \mathrm{id}$. In the rest of the proof we shall make use of the simple fact that

$$\|u\|_{C^\beta_\beta} \le cT^\beta \|u\|_{C^\beta_0},\ u \in C^\beta_0$$

for $\beta \in (0, 1)$. From now on we fix $\beta = 1/2$. Let $w \in \mathbb{B}_{\mathbb{E}_{1,0}}(0, r)$ be given. Analyzing $A^{-1}F = \sum_{i=1}^4 A^{-1}F_i$ term by term we have

$$\left\| \left(\frac{t^2}{s^2}(1 + y^2 s_x^2) - \frac{1}{g^2} \right) \frac{(w + v)_{yy}}{t^2} \right\|_{\mathbb{E}_0} \le \left\| \frac{t^2}{s^2}(1 + y^2 s_x^2) - \frac{1}{g^2} \right\|_{\mathring{\mathbb{E}}_0} \left\| \frac{1}{t^2}(w + v)_{yy} \right\|_{\mathbb{E}_0}$$
$$\le c(s)t^\beta (\|w\|_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1})$$

which entails by Theorem 2.13.

$$\|A^{-1}F_1(w, s)\|_{\mathbb{E}_1} \le c(s)t^\beta (\|w\|_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1})$$

We further have that

$$\|A^{-1}F_2(w, s)\|_{\mathbb{E}_1} = \left\| A^{-1} \frac{g_x}{g}(w + v)_x \right\|_{\mathbb{E}_1} \le c(s)(t\|w\|_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1})$$

by Proposition 2.11 and by

$$\left\| A^{-1} \frac{g_x}{g} w_x \right\|_{\mathbb{E}_1} \le ct^\beta \left\| A^{-1} \frac{g_x}{g} w_x \right\|_{\mathring{\mathbb{E}}_1} \le ct^\beta \|w_x\|_{\mathring{\mathbb{E}}_0} \le ct\|w\|_{\mathbb{E}_1},$$

which follows from

$$w_x \in \mathring{\mathbb{E}}_0 \text{ and } w_x(t, x, y) = \int_0^y w_{xy}(t, x, \eta)\, d\eta.$$

The third term can be estimated as follows

$$\|A^{-1}F_3\|_{\mathbb{E}_1} \le c\|F_3\|_{\mathbb{E}_0} \le c \left\| t2y \frac{s_x}{s} \right\|_{\mathbb{E}_0} (\|w\|_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1})$$
$$\le c(s)t(\|w\|_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1})$$

As for the forth we have

$$\|A^{-1}F_4\|_{\mathbb{E}_1} \le c\|F_4\|_{\mathbb{E}_0} \le \left\| y^2 \frac{t}{s} \left(s_{xx} - \frac{2s_x^2}{s} \right) \frac{(w + v)_y}{t} \right\|_{\mathbb{E}_0}$$
$$\le c(s)t^\beta (\|w\|_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1})$$

We conclude that

$$\|A^{-1}F(w,s)\|_{\mathbb{E}_1} \le c(s)(t^{\beta}\|w\|_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1}) \tag{3.4}$$

By means of Proposition 2.11 $R_N G(w,s,\dot{s})$ can be handled in a similar way as to obtain

$$\|R_N G(w,s,\dot{s})\|_{\mathbb{E}_1} \le c(s)t^{\beta}(\|w\|_{\mathbb{E}_1} + \|v\|_{\mathbb{E}_1}) \tag{3.5}$$

We notice that the constant $c(s)$ only depends on

$$\|s-gt\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{2+\alpha}} \text{ and } \|\dot{s}-g\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{1+\alpha}}$$

and can be thus bounded independently of s in a $\mathcal{S}$-neighborhood of tg. Making T small and r large enough Φ becomes a self-map of any ball in view of (3.4) and (3.5) and this concludes the first part of the proof. We now consider the contractivity. Let $w_1,\ w_2 \in \mathbb{B}_{\mathbb{E}_{1,0}}(0,r)$. Since

$$F(w_1,s) - F(w_2,s) = \left(\frac{1+y^2 s_x^2}{s^2} - \frac{1}{(tg)^2}\right)(w_1-w_2)_{yy} - \frac{g_x}{g}(w_1-w_2)_x$$
$$+\,2y\frac{s_x}{s}(w_1-w_2)_{xy} + y^2\frac{s_{xx}s - 2s_x^2}{s^2}(w_1-w_2)_y$$

and

$$G(w_1,s,\dot{s}) - G(w_2,s,\dot{s}) = \frac{1}{t}[s_x^2\gamma_1(w_1-w_2)_y - ss_x\gamma_1(w_1-w_2)_x]$$

the contractivity is obtained by similar arguments as the self-map property using Propositions 2.9 and 2.11 and the regularity assumption on s. It is also true in this case that the contractivity constant can be made small independently of s in a $\mathcal{S}$-neighborhood of tg by reducing the length of the time interval, if necessary. In conclusion, by Banach contraction principle, we obtain a solution $w \in \mathbb{E}_{1,0}$ of (3.1)–(3.3) and thus a solution u of (2.1)–(2.3). It only remains to prove the additional regularity. As far as u_D and u_N are concerned the claimed regularity follows from Corollary 2.12. As for w it is easily seen that the properties of s imply that

$$F(w,s) = \left(F(w,s) - \frac{g_x^2}{g}\right) + \frac{g_x^2}{g} \in \overset{\circ}{\mathbb{E}}_0 + \mathrm{BUC}^{2+\alpha}$$

and that

$$G(w,s,\dot{s}) \in \mathrm{C}_0^{\beta}\,\mathrm{BUC}^{1+\alpha}$$

for any $w \in \mathbb{E}_{1,0}$. Theorem 2.13 and Propositions 2.9 and 2.11 (see also Remark 2.3) therefore entail that $\Phi(w) \in \mathrm{C}_0^{\beta}\,\mathrm{BUC}^{2+\alpha}$. The reason why $w \notin \overset{\circ}{\mathbb{E}}_1$ lies in the fact that $F(w,s)$ does not satisfy the boundary conditions $(\gamma_0 v_x = g_x)$. □

3.2. *Continuous dependence on the front function*

In the next proposition the dependence of the solution u_s upon the front function s is analyzed.

PROPOSITION 3.2. *Given* $0 < \epsilon < g_0$ *and* $s_1, s_2 \in \mathbb{B}_{\mathbb{S}}(gt, \epsilon)$ *and the corresponding solutions* u_1 *and* u_2 *of* (2.1)–(2.3) *we have that*

$$\|u_1 - u_2\|_{\mathbb{E}_1} \le c\, t^{1/2} \|s_1 - s_2\|_{\mathbb{S}}\,.$$

In other words the solution constructed in the previous theorem depends Lipschitz continuously on s in a $\mathcal{S}$-neighborhood of tg. Moreover, we have that

$$\|u_1 - u_2\|_{\mathrm{C}^{1/2}\,\mathrm{BUC}^{2-k+\alpha}} \le ct^{k/2}(\|u_1 - u_2\|_{\mathbb{E}_1} + \|s_1 - s_2\|_{\mathbb{S}})$$

for $k = 0, 1$.

Proof. Again we set $\beta = 1/2$ and assume, without loss of generality, that $G^* = \mathrm{id}$. It follows from the proof of Theorem 3.1 that a solution to (2.1)–(2.3) exists for any $s \in \mathbb{B}_{\mathbb{S}}(gt, \epsilon)$ and lies in $\mathbb{B}_{\mathbb{E}_{1,0}}(0, r)$ for an appropriate $r > 0$. Let s_1, s_2 be two such functions. Then

$$w_1 - w_2 = \Phi(w_1, s_1) - \Phi(w_2, s_2)$$

which gives

$$\begin{aligned}
&w_1 - w_2 \\
&= A^{-1}\Bigg\{ \left(\frac{1 + y^2 s_{1x}^2}{s_1^2} - \frac{1 + y^2 s_{2x}^2}{s_2^2}\right)(w_{1yy} + v_{yy}) + \left(\frac{1 + y^2 s_{2x}^2}{s_2^2} - \frac{1}{(tg)^2}\right)(w_{1yy} - w_{2yy}) \\
&\quad - \frac{g_x}{g}(w_{1x} - w_{2x}) + 2y\left(\frac{s_{1x}}{s_1} - \frac{s_{2x}}{s_2}\right)(w_1 + v)_{xy} + 2y\frac{s_{2x}}{s_2}(w_1 - w_2)_{xy} \\
&\quad + y^2\left(\frac{s_{1xx}s_1 - 2s_{1x}^2}{s_1^2} - \frac{s_{2xx}s_2 - 2s_{2x}^2}{s_2^2}\right)(w_1 + v)_y \\
&\quad + \frac{s_{2xx}s_2 - 2s_{2x}^2}{s_2^2}(w_1 - w_2)_y \Bigg\} + R_N \frac{1}{t} s\{(s_{1x}^2 - s_{2x}^2)\gamma_1 (w_1 + v)_y + s_{2x}^2 \gamma_1 (w_1 - w_2)_y \\
&\qquad - (s_1 s_{1x} - s_2 s_{2x})\gamma_1 (w_1 + v)_y + s_2 s_{2x}\gamma_1 (w_1 - w_2) + (s_1\dot{s}_1 - s_2\dot{s}_2)\} \\
&= A^{-1}(\mathrm{I{+}II{+}III{+}IV{+}V{+}VI{+}VII}) + R_N(\mathrm{VIII{+}IX{+}X{+}XI{+}XII})
\end{aligned}$$

From this we obtain

$$\begin{aligned}
\|w_1 - w_2\|_{\mathbb{E}_1} &\le c\|\mathrm{I{+}II{+}IV{+}V{+}VI{+}VII}\|_{\mathbb{E}_0} + ct^{\beta}\|w_{1x} - w_{2x}\|_{\mathring{\mathbb{E}}_0} \\
&\quad + c\|\mathrm{VIII{+}IX{+}X{+}XI{+}XII}\|_{\mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{1+\alpha}}
\end{aligned}$$

using Theorem 2.13 and Propositions 2.9 and 2.11. Now similar calculations to those used in the proof of the previous theorem we obtain

$$
\begin{aligned}
&\|\text{I+IV+VI}\|_{\mathbb{E}_0} \le c(\|w\|_{\mathbb{E}_1})t^{\beta}\|s_1 - s_2\|_{\mathbb{S}}\\
&\|\text{II+V+VII}\|_{\mathbb{E}_0} \le c(\|s\|_{\mathbb{S}})t^{\beta}\|w_1 - w_2\|_{\mathbb{E}_1}\\
&\|w_{1x} - w_{2x}\|_{\overset{\circ}{\mathbb{E}}_0} \le c\,t\|w_1 - w_2\|_{\mathbb{E}_1}\\
&\|\text{VIII+X+XII}\|_{\mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{1+\alpha}} \le c(\|w\|_{\mathbb{E}_1})t^{\beta}\|s_1 - s_2\|_{\mathbb{S}}\\
&\|\text{IX+XI}\|_{\mathrm{C}^{\beta}_{\beta}\,\mathrm{BUC}^{1+\alpha}} \le c(\|s\|_{\mathbb{S}})t^{\beta}\|w_1 - w_2\|_{\mathbb{E}_1}.
\end{aligned}
$$

The additional estimates can be obtained observing that

$$\Phi(w_1, s_1) - \Phi(w_2, s_2) = \Phi(w_1 - w_2, s_2) + (\Phi(w_1, s_1) - \Phi(w_1, s_2)).$$

In fact, for $k = 0$, the first term in the sum can be estimated as in the end of the proof of Theorem 3.1. As for $k = 2$ or as for the second term we use the estimates derived in Corollary 2.10 and Proposition 2.11. □

We want now to take a look at the problem solved by s, that is at (2.4)–(2.5), assuming, this time, that u is given. The first of the next two subsections will be devoted to the case $\delta = 0$ and the second to the case $\delta > 0$. The existence of a solution for the full system will be discussed for the two cases separately in the respective subsections.

3.3. *The case* $\delta = 0$

In this subsection we want to analyze the solvability of the following Hamilton-Jacobi equation

$$s_t - \sqrt{1 + s_x^2}\, v = 0 \tag{3.6}$$

$$s(0) \equiv 0 \tag{3.7}$$

for a given function $v \in \mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}$ satisfying $v(0) = g$. We shall first use the method of characteristics to produce a solution and then carefully analyze the ODEs involved to derive the regularity properties we need for our approach to work. The dependence of s on v will also be studied. By means of the standard method of characteristics (cf. [**?**]) it is possible to transform (3.6)–(3.7) into the following system of ODEs:

$$\frac{\partial}{\partial \rho} t = 1, \quad t(0) = 0 \tag{3.8}$$

$$\frac{\partial}{\partial \rho} x = -\frac{q}{\sqrt{1 + q^2}}\, v(t, x), \quad x(0) = r \tag{3.9}$$

$$\frac{\partial}{\partial\rho}\, s = p - \frac{q^2}{\sqrt{1+q^2}}\, v(t,x), \quad s(0) = 0 \tag{3.10}$$

$$\frac{\partial}{\partial\rho}\, p = \sqrt{1+q^2}\, v_t(t,x), \quad p(0) = g(r) \tag{3.11}$$

$$\frac{\partial}{\partial\rho}\, q = \sqrt{1+q^2}\, v_x(t,x), \quad q(0) = 0 \tag{3.12}$$

(3.13)

for positive ρ and $r \in \mathbb{R}$. This system is easily seen to reduce to the single equation

$$\dot{x}(t) = \tanh\left(\int_0^t v_x(\sigma, x(\sigma))\, d\sigma\right) v(t, x(t))\,, \quad x(0) = \beta, \tag{3.14}$$

from which the solution s can be reconstructed. In fact denoting by $X_t \in \mathrm{Diff}^1(\mathbb{R})$ the diffeomorphism obtained by taking the value of the solution of (3.14) at time t for any initial value r, it follows that

$$\begin{aligned} s(t,x) = g(X_t^{-1}(x))t &+ \int_0^t \int_0^\tau \cosh(Y(\sigma)) v_t(\sigma, X_\sigma(X_t^{-1}(x)))\, d\sigma\, d\tau \\ &- \int_0^t \tanh(Y(\tau)) \sinh(Y(\tau)) v(\tau, X_\tau(X_t^{-1}(x)))\, d\tau \end{aligned} \tag{3.15}$$

where we put $Y(\tau) = \int_0^\tau v_x(\sigma, X_\sigma(X_t^{-1}(x))\, d\sigma)$. We also observe that

$$s_x(t,x) = \int_0^t \sinh\left(\int_0^\tau v_x(\sigma, X_\sigma(X_t^{-1}(x)))\, d\sigma\right) d\tau. \tag{3.16}$$

This and (3.7) allows us to avoid dealing directly with (3.15) which would involve the time derivative of v. Analyzing the regularity properties of X as a vector valued function of time we arrive at the following theorem

THEOREM 3.3. *Assume that* $v \in \mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}$ *is given with* $v(0) = g$. *Then there exist a unique local solution* s *of* (3.6)–(3.7) *with*

$$s \in \mathrm{C}^{1+1/2}\,\mathrm{BUC}^{1+\alpha} \cap \mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}\,.$$

Furthermore there is a $T(g) > 0$ *such that the solution* s *exists on* $[0, T(g)]$ *independently of* v *in a neighborhood of* g *in* $\mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}$.

Proof. Set $\beta = 1/2$. Since $v \in \mathrm{C}^\beta\,\mathrm{BUC}^{2+\alpha}$, it follows that (3.14) possesses a unique local solution $x(\cdot, r)$ on some time interval $[0, t_r]$. The regularity of u implies that

$$x(\cdot, r) \in \mathrm{C}^{1+\beta}([0, t_r]).$$

Furthermore, since v is chosen to be in a neighborhood of g we conclude that there exists $t_g > 0$ with $t_r \geq t_g$ for any $r \in \mathbb{R}$. Now $y = \partial_r x$ solves

$$\begin{aligned}\dot{y}(t) &= v(t, x(t)) \tanh\left(\int_0^t v_x(\sigma, x(\sigma))\, d\sigma\right) \int_0^t v_{xx}(\sigma, x(\sigma)) y(\sigma)\, d\sigma \\ &\quad + \tanh\left(\int_0^t v_x(\sigma, x(\sigma))\, d\sigma\right) v_x(t, x(t)) y(t), \\ y(0) &= 1.\end{aligned}$$

It is not difficult to see that there is $\tilde{t}_g > 0$ such that the solution of the above ODE exists on $[0, \tilde{t}_g]$ for any $r \in \mathbb{R}$ and for which

$$y \geq 1/2 \text{ in } [0, \tilde{t}_g] \times \mathbb{R},$$

since $v \in L^\infty \mathrm{BUC}^{2+\alpha}$. The regularity of v thus gives

$$y = [t \mapsto DX_t] \in \mathrm{C}^\beta \mathrm{BUC}^\alpha,$$

which, in its turn, implies that

$$[t \mapsto X_t] \in \mathrm{C}^\beta \mathrm{Diff}^{1+\alpha}$$

and

$$[t \mapsto X_t^{-1}] \in \mathrm{C}^\beta \mathrm{Diff}^{1+\alpha}$$

Finally we see that

$$[t \mapsto (X_t)^*] \in \mathrm{C}^\beta \mathcal{L}(\mathrm{BUC}^{1+\alpha}).$$

and

$$[t \mapsto (X_t)_*] \in \mathrm{C}^\beta \mathcal{L}(\mathrm{BUC}^{1+\alpha}).$$

Thus (3.16) implies that

$$s_x \in \mathrm{C}^\beta \mathrm{BUC}^{1+\alpha}$$

and (3.6) that

$$s_t \in \mathrm{C}^\beta \mathrm{BUC}^{1+\alpha}.$$

Owing to

$$s(t, \cdot) = \int_0^t \sqrt{1 + s_x^2(\tau, \cdot)}\, v(\tau, \cdot)\, d\tau$$

we see that

$$s(t,\cdot) \in \mathrm{BUC}^{\alpha}$$

and conclude that

$$s \in \mathrm{C}^{1+\beta}\,\mathrm{BUC}^{1+\alpha} \cap \mathrm{C}^{\beta}\,\mathrm{BUC}^{2+\alpha}$$

with $s(0) = 0$ and $\dot{s}(0) = \sqrt{1 + s_x^2(0)}\gamma_1 u(0) = g$. For $T(g)$ we take $\min\{t_g, \tilde{t}_g\}$. □

Next we analyze the dependence of the solution s on the datum v.

PROPOSITION 3.4. *Let* $v_1, v_2 \in \mathbb{B}_{\mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}}(g, \delta)$ *and* s_1, s_2 *be the solutions of* (3.6)–(3.7) *with* v *replaced by* v_1 *and* v_2*, respectively. Then*

$$\|s_1 - s_2\|_{\mathbb{S}} \le c(t\,\|v_1 - v_2\|_{\mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}} + \|v_1 - v_2\|_{\mathrm{C}^{1/2}\,\mathrm{BUC}^{1+\alpha}}). \tag{3.17}$$

Proof. Since s_1 and s_2 satisfy (3.6) we only need to estimate $s_{1x} - s_{2x}$. Owing to (3.16)

$$\begin{aligned} s_{1x} - s_{2x} = \int_0^t \Bigg\{ &\sinh\left(\int_0^t v_{1x}(\sigma, X_{1\sigma}(X_{1_t}^{-1}(x)))\,d\sigma\right) \\ &- \sinh\left(\int_0^t v_{2x}\big(\sigma, X_{2\sigma}(X_{2_t}^{-1}(x))\big)\,d\sigma\right)\Bigg\}\,d\tau \end{aligned}$$

where X_{j_t} comes from solving

$$\dot{x} = \tanh\left(\int_0^t v_{j_x}(\sigma, x(\sigma))\right) v_j(t, x)$$

for $j = 1, 2$. Thus, the problem is further reduced to estimating $X_1 - X_2$ for which we easily compute

$$\|[t \mapsto (X_{1t} - X_{2t})^*]\|_{\mathrm{C}^{\beta}\,\mathcal{L}(\mathrm{BUC}^{1+\alpha})} \le c\|v_1 - v_2\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{2+\alpha}}. \tag{3.18}$$

It follows that

$$\|s_{1x} - s_{2x}\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{1+\alpha}} \le ct\|v_1 - v_2\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{2+\alpha}}.$$

Thus the estimate

$$\begin{aligned} &\|s_{1t} - s_{2t}\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{1+\alpha}} \\ &\quad\le \|(\sqrt{1 + s_{1x}^2} - \sqrt{1 + s_{2x}^2})v_1\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{1+\alpha}} + \|\sqrt{1 + s_{2x}^2}(v_1 - v_2)\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{1+\alpha}} \\ &\quad\le ct\|v_1 - v_2\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{2+\alpha}} + c\|v_1 - v_2\|_{\mathrm{C}^{\beta}\,\mathrm{BUC}^{1+\alpha}} \end{aligned}$$

concludes the proof. □

We conclude this subsection with the main existence result. For a definition of the space $\mathbb{E}_1$ we refer to (2.26).

THEOREM 3.5. *Assume that* $g \in \mathrm{BUC}^{3+\alpha}$ *for some* $\alpha \in (0,1)$. *Then system* (2.1)–(2.5) *possesses a unique local solution* $(u,s) \in \mathbb{E}_1 \times \mathbb{S}$ *with*

$$\gamma_1 u \in \mathrm{C}^{1/2}\,\mathrm{BUC}^{2+\alpha}\,.$$

In particular the free boundary problem (1.1)–(1.5) *possesses a unique local classical solution.*

Proof. We define $\Phi_1(s) = u_s = w_s + u_D + u_N$ to be the solution to (2.1)–(2.3) for given s so that $w_s = \Phi(w_s, s)$ for $\Phi = A^{-1}F + R_N G$. Denote by $\Phi_2(v)$ the solution of the Hamilton-Jacobi equation with datum v. Then a solution can be produced from a fixed point s of $\Phi_3 := \Phi_2 \circ \Phi_1$ taking $(\Phi_1(s), s)$. We thus need to prove that Φ_3 is a self map of some complete set and contractive thereon to apply Banach contraction principle. We shall take $\overline{\mathbb{B}}_{\mathbb{S}}(gt, \varepsilon)$ for the complete set where $g_0 > \varepsilon > 0$ is to be chosen appropriately. Given $s \in \overline{\mathbb{B}}_{\mathbb{S}}(gt, \varepsilon)$ we use Theorem 3.1 to produce u_s with $\gamma_1 u_s(0) = g$. Since the modulus of continuity $\gamma_1 u$ can be chosen independent of $s \in \overline{\mathbb{B}}_{\mathbb{S}}(gt, \varepsilon)$ we can use Proposition 3.3 to construct a solution $\Phi_3(s)$ of the Hamilton-Jacobi equation with datum $\gamma_1 u_s$ which lies again in $\overline{\mathbb{B}}_{\mathbb{S}}(gt, \varepsilon)$ and such that its modulus of continuity doesn't depend on u_s and thus on $s \in \overline{\mathbb{B}}_{\mathbb{S}}(gt, \varepsilon)$, either. Thus it is only a question of making the time interval small enough to see that Φ is a self map of $\overline{\mathbb{B}}_{\mathbb{S}}(gt, \varepsilon)$. To see that it is contractive use Propositions 3.2 and 3.4 which give

$$\begin{aligned}
&\|\Phi(s_1) - \Phi(s_2)\|_{\mathbb{S}} \\
&\quad \le ct\|\gamma_1 u_{s_1} - \gamma_1 u_{s_2}\|_{\mathrm{C}^\beta\,\mathrm{BUC}^{2+\alpha}} + c\|\gamma_1 u_{s_1} - \gamma_1 u_{s_2}\|_{\mathrm{C}^\beta\,\mathrm{BUC}^{1+\alpha}} \\
&\quad \le c\,t^{1/2}\|s_1 - s_2\|_{\mathbb{S}},
\end{aligned}$$

and the proof is complete. □

3.4. *The case* $\delta > 0$

The last part of the paper is devoted to the local existence of a regular solution to the free boundary problem with the curvature term. Recall that this is equivalent to choosing $\delta > 0$. As in the previous subsection we shall consider first

$$\dot{s} = \delta \frac{v}{1+s_x^2} s_{xx} + v\sqrt{1+s_x^2}, \tag{3.19}$$

$$s(0,\cdot) \equiv 0, \tag{3.20}$$

for fixed $v \in C^\beta \,\mathrm{BUC}^{2+\alpha}$ and $\beta \in (0, 1)$, and then deal with the whole system. As there is a well developed maximal regularity theory for parabolic problems we shall take advantage of existing results to deal with the above equations. We shall make use of the optimal regularity result formulated in [**?**, Corollary 6.1.6 (iv)] to obtain

THEOREM 3.6. *Given* $v \in V^+ = \{C^\beta \,\mathrm{BUC}^{1+\alpha} \mid v > 0\}$ *there exists a unique solution* s *to* (3.19)–(3.20) *with*

$$s \in C^{1+\beta} \,\mathrm{BUC}^{1+\alpha} \cap C^\beta \,\mathrm{BUC}^{3+\alpha}.$$

Moreover, if s_1 *and* s_2 *are solutions of* (3.19)–(3.20) *to data* v_1 *and* v_2 *then*

$$\|s_1 - s_2\|_{C^{1+\beta} \,\mathrm{BUC}^{1+\alpha} \cap C^\beta \,\mathrm{BUC}^{3+\alpha}} \le c\, \|v_1 - v_2\|_{C^\beta \,\mathrm{BUC}^{1+\alpha}}$$

Proof. Observe that that the above equation has the structure

$$(\partial_t + A(s_x, v))s = v\sqrt{1 + s_x^2}, \; s(0) = 0,$$

where $A(s_x, v)$ is defined in the obvious way. It follows from [**?**, Corollary 6.1.6 (iv)] and from $v \in V^+$ that

$$\partial_t + A(0, v) \in \mathcal{L}_{is}(C^{1+\beta} \,\mathrm{BUC}^{1+\alpha} \cap C_0^\beta \,\mathrm{BUC}^{3+\alpha}, C_0^\beta \,\mathrm{BUC}^{1+\alpha})$$

for any such v. Since the latter operator happens to be the linearization of $\partial_t + A(s_x, v)$ in the initial value we obtain a solution of (3.19)–(3.20) via the implicit function theorem. Since A depends linearly on v we also get the additional estimate. □

We are now in a position to prove the following final existence result. The space $\mathbb{E}_1$ is defined as in (2.26).

THEOREM 3.7. *Assume that* $g \in \mathrm{BUC}^{3+\alpha}$. *Then there is unique local classical solution* $(u, s) \in \mathbb{E}_1 \times \mathbb{S}$ *of* (2.1)–(2.5) *with*

$$\gamma_1 u \in C^{1/2} \,\mathrm{BUC}^{2+\alpha}.$$

Thus the free boundary problem (1.1)–(1.5) *with curvature term possesses a unique classical solution.*

Proof. The proof follows along the same lines that the one of Theorem 3.5. One proves first that $\Phi_3(s) = \Phi_2(\Phi_1(s))$ is a self map of $\mathbb{B}_{\mathbb{S}}(tg, \varepsilon)$ for an appropriate $\varepsilon > 0$, for Φ_i, $i = 1, 2$, defined as in the proof of 3.5. Then, by reducing the interval length, if necessary, it is possible to obtain contractivity by using Theorem 3.6. and Proposition 3.2. □

We conclude the paper with some final remarks.

REMARK 3.1. It is natural to ask the question about maximal or global existence for the solution. A distinction need to be made between the two cases $\delta = 0$ and $\delta > 0$. In the first the characteristics of (3.6)–(3.7) generically cross in finite time. It is not clear if any benefit could come from the coupling to singular BVPs. In the second case it is possible to continue the solution to its maximal existence interval with the same technique used in the above existence proof. The question of global existence remains open.

REMARK 3.2. The results concerning singular BVPs of Section 2 can be generalized to higher dimensions if the domain degeneration only occurs in one direction. Indeed the nature of the symbols involved is not affected in this case. Multidirectional degenerations remain an open problem.

Acknowledgements

The author gratefully acknowledges the support of the Swiss National Science Foundation.

REFERENCES

[1] ALFREY, T., GURNEE, E. F. and Lloyd, W., *Diffussion in glassy polymers*, J. Polymer Sci. Part C, *12* (1966), 249–261.
[2] AMANN, H., *Operator-Valued Fourier Multipliers, Vector-Valued Besov-Spaces and Applications*, Math. Nachr., *186* (1997), 5–56.
[3] AMANN, H., *Linear and Quesilinear Parabolic Problems*, vol II, Birkhäuser, Basel, in preparation.
[4] ASTARITA, G. and SARTI, G. C., *A class of mathematical models for sorption of swelling solvents in glassy polymers*, Polymer Engng. and Sci., *18* (1978), 388–395.
[5] COHEN, D. S. and ERNEUX, T., *Free boundary problems in controlled release pharmaceuticals. I. Diffusion in glassy polymers*, SIAM J. Appl. Math., *48* (1988), 1415–1465.
[6] ESCHER, J. and SIMONETT, G., *Maximal regularity for a free boundary problem*, NoDEA, *2* (1995), 463–510.
[7] ESCHER, J. and SIMONETT, G., *Classical solutions of multidimensional Hele-Shaw models*, SIAM J. MAth. Anal., *28* (1997),1028–1047.
[8] ESCHER, J. and SIMONETT, G., *A, center manifold analysis for the Mullins-Sekerka models*, J. Diff. Eq., *143* (1998), 267–292.
[9] GUIDOTTI, P., *Singular quasilinear abstract Cauchy problems*, Nonlin. Anal. TMA, *32* (1998), 667–695.
[10] GUIDOTTI, P. and PELESKO, J. A., *Transient Instability in Case II Diffusion*, Journal of Polymer Science, Part B: Polymer Physics, *36* (1998), 2941–2947.
[11] JOHN, F., *Partial Differential Equations*, Springer, New York, 1982.
[12] LUNARDI, A., *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.
[13] SCHWARTZ, L., *Théorie des Distributions*, Hermann, Paris, 1966.

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