TWO-PHASE FLOW IN ROTATING HELE-SHAW CELLS WITH CORIOLIS EFFECTS

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Abstract. The free boundary problem of a two phase flow in a rotating Hele-Shaw cell with Coriolis effects is studied. Existence and uniqueness of solutions near spheres is established, and the asymptotic stability and instability of the trivial solution is characterized in dependence on the fluid densities.

1. Introduction

The motion of one or two fluids confined to a narrow gap between two parallel plates is an interesting problem with a long history. It is the classical set up of the so-called Hele-Shaw problem [13]. It is well-known that instability driven pattern formation such as fingering can occur under appropriate assumptions on the viscosity of the fluids [15] and on the surface tension at their interface. There is a vast empirical and theoretical literature concerning this classical problem. From the purely mathematical point of view the problem has been extensively studied in its original formulation as a one [5, 6, 9, 10] and as a two-phase [19] problem and, more recently, also for one fluid in the case of rotating plates [7]. The focus is here on the two-phase problem with rotating plates and including the effects of Coriolis forces. This setting has recently been considered in the physical literature by a variety of authors [1, 11, 18]. It first appeared in [16] where the effects of rotation were introduced in a ad-hoc fashion into Darcy's law for the one-phase problem

$$\nabla p = -\frac{12\eta}{b^2}\vec{v} + \varrho\omega^2\vec{x} + 2\varrho\omega\vec{z}\times\vec{v}\,,$$

where $\vec{z}$ is the axis of rotation, b is the plate spacing, $\vec{x}$ is the two-dimensional position vector, ω is the (constant) angular velocity of the plates, whereas $\vec{v}$, p, η, and ϱ are the fluid's velocity, pressure, viscosity, and density, respectively. The last term accounts for Coriolis' force. While most of the literature hitherto neglected this force, recent studies performed a model derivation for a one-phase Hele-Shaw type model with Coriolis force by means of a standard gap averaging technique starting from Navier-Stokes' equations, cf. [17, 18]. The authors of [18] observe that the effects due to the Coriolis term in their equations can be larger than the inertial terms typically neglected in small Reynolds number type reductions. They also point out the fact that simpler ad-hoc models used earlier, while qualitatively similar, do eventually lead to a different and inaccurate prediction of the growth rate of the unstable modes involved in the fingering phenomenon.

More recently, the effects of a Coriolis term on the fingering patterns in the rotating two-phase Hele-Shaw problem have been studied in [1, 11] where the formal linear stability analyses of [16, 18] for the one-phase problem are extended to the weakly nonlinear case by the use of formal expansions and to the fully nonlinear regime by means of numerical simulations. Previous numerical and experimental results included [4, 17], whereas the practical relevance of the problem is attested by a number of publications cited in [1] for that very purpose.

In this paper local existence of a unique classical solution for nearly circular initial interfaces is established for the general rotating two-phase Hele-Shaw problem with Coriolis effects in the

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formulation proposed in [1, 11], which is, in turn, based on a derivation similar to that of [18] for the single phase case. It leads to the following generalized Darcy's law

$$\nabla P_j = -\alpha_j \vec{v}_j + \beta_j \vec{z} \times \vec{v}_j \,,$$

for the velocities of the inner and outer fluids ($j = \mathrm{i}, \mathrm{o}$), for constants α_j and β_j defined below and for the pressure P_j related to the hydrostatic pressure p_j via

$$P_j = p_j - \frac{\varrho_j \omega^2}{2} |x|^2 \,.$$

The use of the *unusual* pressure P_j yields some useful simplifications as will soon become apparent. Rigorous stability results are also obtained on near circularity assumptions for the initial data by resorting to a general principle of linearized stability. While the circular steady state is exponentially asymptotically stable[1] when the outer fluid is denser, it becomes unstable if this relation is reversed. We shall also see that the equilibrium problems for two phases and one phase are essentially equal. The main techniques used are a transformation of the original free boundary problem to a fixed domain one, a decoupling of the system and a reduction to a single nonlinear and nonlocal evolution for the interface separating the two fluids, and parabolic optimal regularity results in little Hölder spaces.

Related previous mathematical results include the existence, stability, and bifurcation analysis of [7] in the rotating one-phase problem and the global existence and stability of smooth solutions obtained by [19] for nearly circular initial interfaces in the non-rotating two-phase case.

2. Governing Equations and Main Results

We first give a short justification for the mathematical formulation of the physical problem of a rotating Hele-Shaw cell including Coriolis effects. Further details regarding the modeling can be found in [1, 11]. We then end this section with the main results on existence of solutions to the corresponding governing equations and stability properties of the trivial solution.

2.1. Governing Equations.

Consider a circular Hele-Shaw cell of radius $R \geq 2$ and very small gap width $b > 0$, rotating clockwise around the z-axis with a constant angular velocity $\omega > 0$. The cell is assumed to contain two immiscible, incompressible, viscous fluids with densities $\varrho_j > 0$ and viscosities $\eta_j > 0$, where $j =$i labels the inner and $j =$o the outer fluid, respectively. The surface tension between the two fluids is given by $\sigma > 0$. The rotating coordinate system is defined in such a way that its origin is located at the cell center and that rotation is perpendicular to the vector $\vec{z} = (0, 0, 1)$. Let $\Omega^j(t)$ denote the region of space occupied by fluid $j \in \{\mathrm{i,o}\}$ at time t, and let $\Gamma(t)$ denote the sharp interface separating the two fluids. The unit normal vector $\nu_{\Gamma(t)}$ to $\Gamma(t)$ is assumed to point from $\Omega^{\mathrm{i}}(t)$ to $\Omega^{\mathrm{o}}(t)$.

The basic hydrodynamic equation of the system is a generalized *Darcy's law* relating the pressure fields $P_j = P_j(t)$ to the two-dimensional gap-averaged flow velocities $\vec{v}_j = \vec{v}_j(t)$ through

$$\nabla P_j = -\alpha_j \vec{v}_j + \beta_j (\vec{z} \times \vec{v_j}) \quad \text{in} \quad \Omega^j(t) \,. \tag{2.1}$$

The numbers

$$\alpha_j := \frac{12\eta_j}{b^2} E_j \,, \qquad \beta_j := \frac{12\eta_j}{b^2} F_j$$

depend linearly on the Coriolis force terms $E_j > 0$ and $F_j \geq 0$, respectively. The latter, in turn, depend on the rotational Reynolds number $Re_j = \varrho_j \omega b^2 / 12\eta_j$ (see [1]). If Coriolis effects are

[1] Our analytical approach reveals a precise description of the exponential decay in terms of the physical parameters, in particular of the Coriolis terms, see Section 4.

neglected, then $E_j = 1$ and $F_j = 0$, so (2.1) reduces to the usual Darcy's law. Throughout this paper we shall assume that $\alpha_j > 0$ and $\beta_j \geq 0$. Incompressibility of the fluids is expressed by

$$\mathrm{div}\vec{v}_j = 0 \quad \text{in} \quad \Omega^j(t)\,, \tag{2.2}$$

while the interface dynamics is governed by the normal stress balance

$$P_\mathrm{i} - P_\mathrm{o} = \sigma\kappa_{\Gamma(t)} + (\gamma_\mathrm{o} - \gamma_\mathrm{i})|x|^2 \quad \text{on} \quad \Gamma(t)\,, \tag{2.3}$$

and the kinematic boundary condition

$$V = \vec{v}_\mathrm{i}\cdot\nu_\Gamma = \vec{v}_\mathrm{o}\cdot\nu_\Gamma \quad \text{on} \quad \Gamma(t)\ . \tag{2.4}$$

Here, $\gamma_j := \varrho_j\omega^2/2$ whereas $\kappa_{\Gamma(t)}$ denotes the curvature of $\Gamma(t)$ which is taken to be positive if $\Omega^\mathrm{i}(t)$ is convex. The normal velocity of the interface is given by $V = V(t)$. On the outer boundary of the cell a no-slip condition

$$\vec{v}_\mathrm{o}\cdot x = 0 \quad \text{on} \quad [|x| = R] \tag{2.5}$$

is imposed.

Observe that the inner and outer fluids' masses are conserved since, by Reynold's transport theorem, one has that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\mathrm{o}(t)} \mathrm{d}x = -\int_{\Gamma(t)} \vec{v}_\mathrm{o}\cdot\nu_\Gamma\,\mathrm{d}\sigma + \int_{[|x|=R]} \vec{v}_\mathrm{o}\cdot\frac{x}{R}\,\mathrm{d}\sigma$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^\mathrm{i}(t)} \mathrm{d}x = \int_{\Gamma(t)} \vec{v}_\mathrm{i}\cdot\nu_\Gamma\,\mathrm{d}\sigma\,,$$

which are both zero in view of Gauss' Theorem and (2.2). Since $\mathrm{rot}(\vec{z}\times\vec{v}_j) = (0, 0, \mathrm{div}\vec{v}_j)$ we also note that the flows are irrotational in the bulk, i.e. $\mathrm{rot}\vec{v}_j = 0$ in $\Omega^j(t)$, due to (2.1) and (2.2).

2.2. **The System in Terms of P_j.** The governing equations in terms of P_j are obtained by taking the divergence on both sides of (2.1) which yields

$$\Delta P_j = 0 \quad \text{in} \quad \Omega^j(t)\,,$$

making use of (2.2) and observing that $\mathrm{rot}\vec{v}_j = 0$ implies $\mathrm{div}(\vec{z}\times\vec{v}_j) = 0$. As for the kinematic boundary condition (2.4), we solve (2.1) for $\vec{v}_j$ to get

$$\vec{v}_j = \frac{1}{|\Theta_j|^2}\big(-\alpha_j\nabla P_j - \beta_j(\vec{z}\times\nabla P_j)\big)$$

with complex numbers $\Theta_j := \alpha_j + i\beta_j$. Then, defining the tangent vector $\tau_\Gamma := -(\vec{z}\times\nu_\Gamma)$ to Γ, we obtain

$$\vec{v}_j\cdot\nu_\Gamma = \frac{1}{|\Theta_j|^2}\big(-\alpha_j\partial_{\nu_\Gamma}P_j - \beta_j\partial_{\tau_\Gamma}P_j\big)\ .$$

Whence (2.4) and (2.5) become

$$-V = \frac{1}{|\Theta_\mathrm{i}|^2}\big(\alpha_\mathrm{i}\partial_{\nu_\Gamma}P_\mathrm{i} + \beta_\mathrm{i}\partial_{\tau_\Gamma}P_\mathrm{i}\big) = \frac{1}{|\Theta_\mathrm{o}|^2}\big(\alpha_\mathrm{o}\partial_{\nu_\Gamma}P_\mathrm{o} + \beta_\mathrm{o}\partial_{\tau_\Gamma}P_\mathrm{o}\big) \qquad \text{on} \quad \Gamma(t)\,,$$

and,

$$\alpha_\mathrm{o}\partial_\nu P_\mathrm{o} + \beta_\mathrm{o}\partial_\tau P_\mathrm{o} = 0 \qquad \text{on} \quad [|x| = R]\,,$$

respectively, where ν denotes the unit outer normal vector and τ the corresponding tangential vector on the cell boundary $[|x|=R]$. Therefore, we arrive at the following free boundary problem for the pressures P_j

$$\Delta P_j = 0 \qquad \text{in } \Omega^j(t)\,, \qquad j=\mathrm{i,o}\,, \tag{2.6}$$

$$P_\mathrm{i} - P_\mathrm{o} = \sigma\kappa_{\Gamma(t)} + (\gamma_\mathrm{o}-\gamma_\mathrm{i})|x|^2 \qquad \text{on } \Gamma(t)\,, \tag{2.7}$$

$$-V = \frac{1}{|\Theta_\mathrm{i}|^2}\big(\alpha_\mathrm{i}\partial_{\nu_\Gamma}P_\mathrm{i} + \beta_\mathrm{i}\partial_{\tau_\Gamma}P_\mathrm{i}\big) = \frac{1}{|\Theta_\mathrm{o}|^2}\big(\alpha_\mathrm{o}\partial_{\nu_\Gamma}P_\mathrm{o} + \beta_\mathrm{o}\partial_{\tau_\Gamma}P_\mathrm{o}\big) \quad \text{on } \Gamma(t)\,, \tag{2.8}$$

$$\alpha_\mathrm{o}\partial_\nu P_\mathrm{o} + \beta_\mathrm{o}\partial_\tau P_\mathrm{o} = 0 \qquad \text{on } [|x|=R]\,, \tag{2.9}$$

for $t>0$, complemented with an initial surface $\Gamma(0)=\Gamma^0$. Since only derivatives and the difference of the two pressures enter the system, uniqueness can only be expected up to additive constants for them. As we shall later see, this is one of the obstacles that need to be overcome from a mathematical viewpoint.

2.3. **Main Theorems.** To give a precise formulation of our mathematical results on (2.6)-(2.9), we parametrize the boundary $\Gamma(t)$ over the unit sphere $\mathbb{S}^1 := \{x\in\mathbb{R}^2\,;\,|x|=1\}$. To this end we introduce, for $s\ge 0$, so-called little Hölder spaces $h^s(U)$ over an open subset U of $\mathbb{R}^n$ as the closure of $BUC^\infty(U)$ in $BUC^s(U)$. Here $BUC^s(U)$ consists of all functions $f: U\to\mathbb{R}$ with bounded and uniformly continuous derivatives up to order $[s]$ and with uniformly $(s-[s])$-Hölder continuous derivatives of order $[s]$. If M is a (sufficiently smooth) submanifold of $\mathbb{R}^n$, we define $h^s(M)$ by means of an atlas for M in the canonical way. In the following we shall identify a function $h:\mathbb{S}^1\to\mathbb{R}$ with the function $\bar h:[0,2\pi]\to\mathbb{R}$ given by $\bar h(\theta) := h\big(e^{i\theta}\big)$ for $0\le\theta\le 2\pi$. The bar will often be dropped as no confusion seems likely.

Fix $a\in(0,1/4)$ and $\delta\in(0,1)$ and set

$$\mathcal{V} := \{\rho\in h^{4+\delta}(\mathbb{S}^1)\,;\,\|\rho\|_\infty < a\}\,,$$

and, for $\rho\in\mathcal{V}$,

$$\Omega_\rho^\mathrm{i} := \left\{x\in\mathbb{R}^2\setminus\{0\}\,;\,|x|<1+\rho\left(\frac{x}{|x|}\right)\right\}\cup\{0\}\,, \quad \Omega_\rho^\mathrm{o} := \Omega\setminus\overline{\Omega_\rho^\mathrm{i}}$$

with $\Omega := \mathbb{B}(0,R)$ where, we recall, $R\ge 2$. Then

$$\Gamma_\rho := \left\{x\in\mathbb{R}^2\,;\,|x|=1+\rho\left(\frac{x}{|x|}\right)\right\} = \Big\{\big[1+\rho(y)\big]y\,;\,y\in\mathbb{S}^1\Big\}$$

separates the domains Ω_ρ^i and Ω_ρ^o. Since Γ_ρ can be described as the zero level set of

$$N_\rho(x) := |x|-1-\rho\left(\frac{x}{|x|}\right)\,, \quad 3/4<|x|<5/4\,,$$

with $N_\rho<0$ in $\Omega_\rho^\mathrm{i}\cap[3/4<|x|<5/4]$, the unit normal $\nu_\rho(x)$ at $x\in\Gamma_\rho$ pointing from Ω_ρ^i to Ω_ρ^o is given by

$$\nu_\rho(x) = \frac{\nabla N_\rho(x)}{|\nabla N_\rho(x)|}\ . \tag{2.10}$$

In the following, we let $\tau_\rho = -(\vec{z}\times\nu_\rho)$ denote the corresponding tangential vector. Next, suppose that $\rho\in C\big([0,T],\mathcal{V}\big)\cap C^1\big([0,T],h^{1+\delta}(\mathbb{S}^1)\big)$ for some $T>0$ and set

$$N_\rho(t,x) := N_{\rho(t)}(x) = |x|-1-\rho\left(t,\frac{x}{|x|}\right)\,, \quad t\in[0,T]\,, \qquad 3/4<|x|<5/4\ .$$

Observe that the normal velocity V_ρ of the moving boundary Γ_ρ equals $-\partial_t N_\rho/|\nabla N_\rho|$. Consequently, if $\rho \in C\big([0,T],\mathcal{V}\big) \cap C^1\big([0,T],h^{1+\delta}(\mathbb{S}^1)\big)$ describes the evolution of the boundary, then (2.6)-(2.9) read as

$$\begin{aligned}
\Delta P_j &= 0 && \text{in } \Omega^j_{\rho(t)}\,, \qquad j=\mathrm{i,o}\,, && (2.11)\\
P_\mathrm{i} - P_\mathrm{o} &= \sigma\kappa_{\Gamma_{\rho(t)}} + (\gamma_\mathrm{o}-\gamma_\mathrm{i})\Big[1+\rho\big(t,\frac{x}{|x|}\big)\Big]^2 && \text{on } \Gamma_{\rho(t)}\,, && (2.12)\\
\partial_t\rho\big(t,\frac{x}{|x|}\big) &= \frac{-1}{|\Theta_j|^2}\big[\alpha_j\nabla P_j + \beta_j(\vec{z}\times\nabla P_j)\big]\cdot\nabla N_{\rho(t)} && \text{on } \Gamma_{\rho(t)}\,, \qquad j=\mathrm{i,o}\,, && (2.13)\\
\alpha_\mathrm{o}\partial_\nu P_\mathrm{o} + \beta_\mathrm{o}\partial_\tau P_\mathrm{o} &= 0 && \text{on } \partial\Omega = R\,\mathbb{S}^1\,, && (2.14)
\end{aligned}$$

for $t>0$ and with $\Gamma_{\rho(0)} = \Gamma^0$. We call a triple $(\rho, P_\mathrm{i}, P_\mathrm{o})$ a *(classical Hölder) solution to* (2.11)-(2.14) provided

$$\rho \in C\big([0,T],\mathcal{V}\big) \cap C^1\big([0,T],h^{1+\delta}(\mathbb{S}^1)\big)$$

for some $T>0$ and

$$P_j(t,\cdot) \in h^{2+\delta}\big(\Omega^j_{\rho(t)}\big)\,, \quad t\in[0,T]\,, \quad j=\mathrm{i,o}\,,$$

satisfy (2.11)-(2.14) pointwise. The main existence result is stated in the following theorem, a proof of which is given in Section 3:

Theorem 2.1. *Let $\delta\in(0,1)$. There is an open zero neighborhood $\mathcal{O}$ in $h^{4+\delta}(\mathbb{S}^1)$ such that for each initial geometry $\rho_0\in\mathcal{O}$, there exist a time $T := T(\rho_0) > 0$ and a classical solution $(\rho, P_\mathrm{i}, P_\mathrm{o})$ to* (2.11)-(2.14) *exists on the interval $[0,T]$ with $\rho(0)=\rho_0$. This solution is unique except for additive constants in the pressures P_i and P_o.*

We also investigate stability properties of the trivial solution corresponding to the geometry $\rho=0$, that is, to the unit circle $\mathbb{S}^1$, and constant pressures $P_\mathrm{i} = c+\sigma+\gamma_\mathrm{o}-\gamma_\mathrm{i}$ and $P_\mathrm{o}=c$ with arbitrary $c\in\mathbb{R}$. The stability properties of the trivial solution are determined by the relative density of the fluids.

Theorem 2.2. *If $\varrho_\mathrm{o} > \varrho_\mathrm{i}$ then the trivial solution $(\rho,P_\mathrm{i},P_\mathrm{o}) = \big(0, c+\sigma+\gamma_\mathrm{o}-\gamma_\mathrm{i}, c\big)$ is locally asymptotically stable. However, if $\varrho_\mathrm{i} > \varrho_\mathrm{o}$, then the trivial solution $(\rho,P_\mathrm{i},P_\mathrm{o}) = \big(0, c+\sigma+\gamma_\mathrm{o}-\gamma_\mathrm{i}, c\big)$ is unstable.*

We refer to Section 4 for more precise statements of the stability results and their proofs. Note that Theorem 2.2 implies that there are no nontrivial equilibrium solutions near the sphere if $\varrho_\mathrm{o} > \varrho_\mathrm{i}$. Equilibria in general are characterized by the following result:

Proposition 2.3. *If $(\Gamma, P_\mathrm{i}, P_\mathrm{o})$ is any equilibrium for* (2.6)-(2.9) *with $\Gamma\in C^{2+\delta}$ and $P_j\in h^{2+\delta}\big(\Omega^j\big)$ for $j=\mathrm{i,o}$, then P_j is constant for $j=\mathrm{i,o}$. Moreover, if $\Gamma=\Gamma_\rho$ for some $\rho\in\mathcal{V}$ and $c := P_\mathrm{i}-P_\mathrm{o}\in\mathbb{R}$, then $\rho\in C^\infty(\mathbb{S}^1)$ satisfies*

$$c = \gamma\frac{(1+\rho)^2 + 2\dot\rho^2 - (1+\rho)\ddot\rho}{\big[(1+\rho)^2+\dot\rho^2\big]^{3/2}} + (\zeta_\mathrm{o}-\zeta_\mathrm{i})(1+\rho)^2 \quad \text{on } \mathbb{S}^1\ . \tag{2.15}$$

This statement is proved in Section 5. Equation (2.15) for the geometry determines equilibria completely. Moreover, the fact that equilibria occur only if the pressures are both constant has a number of implications. In particular, equilibria for the two-phase problem coincide with the equilibria for the one-phase problem (with the fluid inside). More precisely, the one-phase equilibrium problem (as treated e.g. in [8]) may be regarded as a two-phase equilibrium problem by taking $P_\mathrm{o}=0$ and $\varrho_\mathrm{o}=0$. Conversely, the two-phase equilibrium problem is equal to the one-phase equilibrium problem by taking $P_\mathrm{o} = const$. Furthermore, as the pressures at equilibria are necessarily constant, Coriolis force has no influence on the existence of equilibria. Therefore, there are no equilibria for (2.6)-(2.9) if $\varrho_\mathrm{o}\geq\varrho_\mathrm{i}$ as noted above (the case $\varrho_\mathrm{o}=\varrho_\mathrm{i}$ corresponds to considering just one fluid of density zero) while the case $\varrho_{\rm o} < \varrho_{\rm i}$ has been investigated in [8]. The unstable nontrivial equilibria constructed therein[2] by means of bifurcation theory are also equilibria for (2.6)-(2.9).

3. Proof of Theorem 2.1

The proof of Theorem 2.1 is best carried out in a coordinate system in which the moving interface between the liquids is fixed. We therefore begin with the transformation to a fixed domain formulation.

3.1. **An Equivalent Problem on Fixed Domains.** We transform the free boundary problem to fixed domains using the standard Hanzawa-transform. Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function with $\varphi(r) = 1$ for $|r| \le a$ and $\varphi(r) = 0$ for $|r| \ge 3a$ and $\|\varphi'\|_\infty < 1/a$. We introduce a family of $C^{4+\delta}$-diffeomorphisms

$$\phi_\rho \in \mathrm{Diff}^{4+\delta}\big(\mathbb{R}^2, \mathbb{R}^2\big) \cap \mathrm{Diff}^{4+\delta}\big(B^{\rm i}, \Omega_\rho^{\rm i}\big) \cap \mathrm{Diff}^{4+\delta}\big(B^{\rm o}, \Omega_\rho^{\rm o}\big)\,, \quad \rho \in \mathcal{V}\,,$$

where

$$B^{\rm i} := \mathbb{B}(0,1)\,, \quad B^{\rm o} := \Omega \setminus \overline{B^{\rm i}}\,,$$

by setting

$$\phi_\rho(x) := \begin{cases} \Big(|x| + [|x|-1]\rho\big(\frac{x}{|x|}\big)\Big)\frac{x}{|x|} & \text{if } 0 < |x| < 2\,, \\ x & \text{else.} \end{cases}$$

Note that ϕ_ρ maps $\mathbb{S}^1$ onto Γ_ρ. Given $\rho \in \mathcal{V}$, let

$$\phi_\rho^* : BUC(\Omega_\rho^j) \to BUC(B^j)\,, \quad p \mapsto p \circ \phi_\rho$$

denote the push-forward operator and

$$\phi_*^\rho : BUC(B^j) \to BUC(\Omega_\rho^j)\,, \quad q \mapsto q \circ \phi_\rho^{-1}$$

the pull-back operator induced by ϕ_ρ. Given $\rho \in \mathcal{V}$, the transformed differential and boundary operators acting on $Q_j := \phi_\rho^* P_j$ $j = {\rm i}, {\rm o}$ are given by

$$\mathcal{A}_j(\rho) Q_j := \phi_\rho^*\big(\Delta(\phi_*^\rho Q_j)\big)$$

and by

$$\mathcal{B}_j(\rho) Q_j := \frac{-1}{|\Theta_j|^2}\left[\alpha_j\big(\phi_\rho^*\nabla(\phi_*^\rho Q_j)\big) + \beta_j\big(\vec{z} \times \big(\phi_\rho^*\nabla(\phi_*^\rho Q_j)\big)\big)\right] \cdot \big(\phi_\rho^* \nabla N_\rho\big)\,.$$

Defining

$$\mathcal{K}(\rho) := \sigma \phi_\rho^* \kappa_{\Gamma_\rho} + (\gamma_{\rm o} - \gamma_{\rm i})|1+\rho|^2$$

for $\rho \in \mathcal{V}$ and

$$\mathcal{B}_{\rm o} := \alpha_{\rm o}\partial_\nu + \beta_{\rm o}\partial_\tau \quad \text{on} \quad \partial\Omega = R\mathbb{S}^1\,,$$

the free boundary problem (2.11)-(2.14) is transformed to the following problem on fixed domains:

$$\mathcal{A}_j(\rho) Q_j = 0 \qquad \text{in } B^j\,, \quad j = \text{i,o}\,, \tag{3.1}$$

$$Q_{\rm i} - Q_{\rm o} = \mathcal{K}(\rho) \qquad \text{on } \mathbb{S}^1\,, \tag{3.2}$$

$$\partial_t \rho = \mathcal{B}_j(\rho) Q_j \qquad \text{on } \mathbb{S}^1\,, \quad j = \text{i,o}\,, \tag{3.3}$$

$$\mathcal{B}_{\rm o} Q_{\rm o} = 0 \qquad \text{on } R\mathbb{S}^1\,. \tag{3.4}$$

We call a triple $(\rho, Q_{\rm i}, Q_{\rm o})$ a *(classical Hölder) solution to* (3.1)-(3.4) provided

$$\rho \in C\big([0,T], \mathcal{V}\big) \cap C^1\big([0,T], h^{1+\delta}(\mathbb{S}^1)\big)$$

[2]In [8] the fluid is assumed to have density $\varrho = 1$. So to compare it with our results one has to replace ω^2 by $\omega^2\varrho$ therein.

for some $T > 0$ and

$$Q_j(t,\cdot) \in h^{2+\delta}\big(B^j\big)\,, \quad t \in [0,T]\,, \quad j = \mathrm{i},\mathrm{o}\,,$$

satisfy (3.1)-(3.4) pointwise. With this definition, problems (2.11)-(2.14) and (3.1)-(3.4) are equivalent as follows from the next proposition.

Proposition 3.1. *Let $Q_j = \phi_\rho^* P_j$ and $P_j = \phi_*^\rho Q_j$. Then $(\rho, P_\mathrm{i}, P_\mathrm{o})$ is a solution to (2.11)-(2.14) if and only if $(\rho, Q_\mathrm{i}, Q_\mathrm{o})$ is a solution to (3.1)-(3.4).*

Proof. Due to the above definitions of the differential operators, we merely need to ascertain that no regularity loss occurs in the process, i.e., that $P_j \in h^{2+\delta}(\Omega_\rho^j)$ implies $Q_j = \phi_\rho^* P_j \in h^{2+\delta}(B^j)$ for $\rho \in \mathcal{V}$ and vice versa. This, however, easily follows from the fact that $\phi_\rho^* P_j \in BUC^4(B^j)$ when $P_j \in BUC^\infty(\Omega_\rho^j)$ and $\rho \in \mathcal{V} \subset h^{4+\delta}(\mathbb{S}^1)$ and the observation that $h^{2+\delta}(B^j)$ coincides with the closure of $BUC^4(B^j)$ in $BUC^{2+\delta}(B^j)$. □

Next we collect some regularity properties of the operators defined above.

Lemma 3.2. *(i) The operator $\mathcal{A}_j(\rho) \in \mathcal{L}\big(h^{2+\delta}(B^j), h^\delta(B^j)\big)$ is uniformly elliptic and analytic in $\rho \in \mathcal{V}$ for $j = \mathrm{i},\mathrm{o}$.*
(ii) The operator $\mathcal{B}_j(\rho) \in \mathcal{L}\big(h^{2+\delta}(B^j), h^{1+\delta}(\mathbb{S}^1)\big)$ is analytic in $\rho \in \mathcal{V}$ for $j = \mathrm{i},\mathrm{o}$.

Proof. (i) Given $\rho \in \mathcal{V}$, uniform ellipticity of $\mathcal{A}_j(\rho)$ is a consequence of its symbol being

$$a(\rho)(\xi) = \phi_\rho^*\big(|J(\phi_\rho^{-1})\xi|^2 + \Delta\phi_\rho^{-1}\cdot\xi\big)\,, \quad \xi \in \mathbb{R}^2\,,$$

where the Jacobian $J(\phi_\rho^{-1})$ of ϕ_ρ^{-1} satisfies $|J(\phi_\rho^{-1})\xi|^2 \geq \epsilon|\xi|^2$ for some $\epsilon > 0$. Analyticity in ρ follows from the analyticity of ϕ_ρ and of $J(\phi_\rho^{-1}) = \phi_\rho^* J(\phi_\rho)$. For details we refer to [8, § 3.2].

(ii) Note that

$$\nabla\rho\left(\frac{x}{|x|}\right) = \rho'\left(\frac{x}{|x|}\right)\frac{1}{|x|^2}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}\,, \quad x = (x_1,x_2) \in \mathbb{R}^2\setminus\{0\}\,, \quad \rho \in C^1(\mathbb{S}^1)\,. \tag{3.5}$$

Hence, for $\rho \in \mathcal{V}$ and $\phi_\rho = (\phi_\rho^1, \phi_\rho^2)$,

$$\big(\phi_\rho^*\nabla N_\rho\big)(y) = \frac{\phi_\rho(y)}{|\phi_\rho(y)|} - \rho'\left(\frac{\phi_\rho(y)}{|\phi_\rho(y)|}\right)\frac{1}{|\phi_\rho(y)|^2}\begin{pmatrix}-\phi_\rho^2(y)\\ \phi_\rho^1(y)\end{pmatrix}\,, \quad y \in \mathbb{S}^1\,,$$

showing that $\big[\rho \mapsto \phi_\rho^*\nabla N_\rho\big] : \mathcal{V} \to h^{3+\delta}(\mathbb{S}^1)$ is analytic. The definition of $\mathcal{B}_j(\rho)$ entails its analytic dependence on ρ since

$$\phi_\rho^*\nabla(\phi_*^\rho Q_j) = \nabla Q_j\, \phi_\rho^* J(\phi_\rho^{-1})\,.$$

□

The curvature operator $\mathcal{K}$ can easily be computed explicitly.

Lemma 3.3. *The operator $\mathcal{K} : \mathcal{V} \to h^{2+\delta}(\mathbb{S}^1)$ is analytic and given by*

$$\mathcal{K}(\rho) = \sigma\frac{(1+\rho)^2 + 2\dot\rho^2 - (1+\rho)\ddot\rho}{\big[(1+\rho)^2 + \dot\rho^2\big]^{3/2}} + (\gamma_\mathrm{o} - \gamma_\mathrm{i})(1+\rho)^2\,, \quad \rho \in \mathcal{V}\,,$$

and

$$\partial\mathcal{K}(0)[h] = \sigma(-h - \ddot h) + 2(\gamma_\mathrm{o} - \gamma_\mathrm{i})h\,, \quad h \in h^{4+\delta}(\mathbb{S}^1)\,.$$

Proof. If $\rho \in \mathcal{V}$, a parametrization of Γ_ρ is given by

$$(1+\rho(\theta))e^{i\theta} =: x(\theta) + iy(\theta)\,, \quad \theta \in [0,2\pi]\,.$$

Hence the claim is a consequence of the curvature formula

$$\kappa_{\Gamma_\rho} = \frac{\dot x\ddot y - \ddot x\dot y}{\big(\dot x^2 + \dot y^2\big)^{3/2}}$$

and the definition of $\mathcal{K}(\rho)$. □

3.2. **Local Well-Posedness.** In order to establish local existence and uniqueness of solutions, we split (3.1)-(3.4) in three subproblems: two elliptic problems for the transformed pressures and an evolution equation for the geometry. More precisely, given $\rho \in \mathcal{V}$ and a function h defined on $\mathbb{S}^1$, we shall first look for solutions $Q_{\mathrm{i}} = S(\rho, h)$ to the following elliptic problem on B^{i} with Neumann type boundary condition

$$\mathcal{A}_{\mathrm{i}}(\rho)Q_{\mathrm{i}} = 0 \qquad \text{in } B^{\mathrm{i}}\,, \tag{3.6}$$

$$\mathcal{B}_{\mathrm{i}}(\rho)Q_{\mathrm{i}} = h - \frac{\big|\phi_\rho^*(\nabla N_\rho)\big|}{|\Gamma_\rho|}\int_{\Gamma_\rho}\frac{\phi_*^\rho h}{|\nabla N_\rho|}\,\mathrm{d}\sigma \qquad \text{on } \mathbb{S}^1\,, \tag{3.7}$$

$$\int_{\mathbb{S}^1} Q_{\mathrm{i}}\,\mathrm{d}\sigma = \int_{\mathbb{S}^1} h\,\mathrm{d}\sigma\,, \tag{3.8}$$

where $|\Gamma_\rho|$ stands for the length of the closed curve Γ_ρ. In the second step, we shall study the elliptic problem on the annulus B^{o} with Dirichlet-Neumann boundary conditions

$$\mathcal{A}_{\mathrm{o}}(\rho)Q_{\mathrm{o}} = 0 \qquad \text{in } B^{\mathrm{o}}\,, \tag{3.9}$$

$$Q_{\mathrm{o}} = g \qquad \text{on } \mathbb{S}^1\,, \tag{3.10}$$

$$\mathcal{B}_{\mathrm{o}}Q_{\mathrm{o}} = 0 \qquad \text{on } R\,\mathbb{S}^1\,, \tag{3.11}$$

with solution $Q_{\mathrm{o}} = T(\rho, g)$ depending on $\rho \in \mathcal{V}$ and g on $\mathbb{S}^1$. Finally we use the solution operators S and T to derive the evolution equation

$$\partial_t\rho = \mathcal{B}_{\mathrm{o}}(\rho)T\big(\rho, S(\rho, \partial_t\rho) - \mathcal{K}(\rho)\big) \tag{3.12}$$

for ρ. Note that $\partial_t\rho$ appears on both sides of (3.12). As (3.6)-(3.7) is a Neumann-type problem, an additional integral term is introduced on the right hand of side (3.7) which makes the mean zero and thus ensures solvability of the problem. The third equation (3.8) guarantees uniqueness. Let us point out that, provided we can solve (3.12) for $\rho = \rho(t)$, the particular form of the boundary operator $\mathcal{B}_{\mathrm{o}}(\rho)$ and the properties of $T(\rho, g)$ yield (see (3.44) below)

$$\int_{\Gamma_\rho}\frac{\phi_*^\rho\partial_t\rho}{|\nabla N_\rho|}\,\mathrm{d}\sigma = 0\ .$$

Thus the integral term in (3.7) vanishes for $h = \partial_t\rho$ and we may take

$$Q_{\mathrm{i}}(t) = S\big(\rho(t), \partial_t\rho(t)\big)\,, \qquad Q_{\mathrm{o}}(t) = T\big(\rho(t), Q_{\mathrm{i}}(t) - \mathcal{K}(\rho(t))\big)$$

to obtain a solution for the original equations (3.1)-(3.4). We shall be more specific at the end of this section.

In the following two propositions we study the solvability of the elliptic problems.

Proposition 3.4. *Given $\rho \in \mathcal{V}$ and $h \in h^{1+\delta}(\mathbb{S}^1)$,* (3.6)-(3.8) *possesses a unique solution*

$$Q_{\mathrm{i}} = S(\rho, h) \in h^{2+\delta}(B^{\mathrm{i}})\,,$$

and the map

$$[\rho \mapsto S(\rho, \cdot)] : \mathcal{V} \to \mathcal{L}\big(h^{1+\delta}(\mathbb{S}^1), h^{2+\delta}(B^{\mathrm{i}})\big)$$

is analytic.

To prove this statement on the fixed domain B^{i} we shall first prove an auxiliary result on the domain Ω_ρ^{i}:

Lemma 3.5. *Let $\rho \in \mathcal{V}$ and let $\nu = \nu_{\Gamma_\rho}$ and $\tau = \tau_{\Gamma_\rho}$ denote the corresponding outer unit normal and unit tangential vectors to Γ_ρ, respectively. Set*

$$\mu := \frac{-1}{|\Theta_{\rm i}|^2}\big(\alpha_{\rm i}\nu + \beta_{\rm i}\tau\big)$$

and define

$$\mathbb{A} : h^{2+\delta}(\Omega^{\rm i}_\rho) \to h^\delta(\Omega^{\rm i}_\rho) \times h^{1+\delta}(\Gamma_\rho)\,, \quad u \mapsto \big(-\Delta u, \partial_\mu u\big)\ .$$

Then $\ker(\mathbb{A}) = \mathbb{R}\cdot\mathbf{1}$ *and*

$$\mathrm{im}(\mathbb{A}) = \left\{(f,g) \in h^\delta(\Omega^{\rm i}_\rho) \times h^{1+\delta}(\Gamma_\rho)\,;\ \int_{\Omega^{\rm i}_\rho} f\,{\rm d}x = \frac{|\Theta_{\rm i}|^2}{\alpha_{\rm i}}\int_{\Gamma_\rho} g\,{\rm d}\sigma\right\}\ .$$

Proof. Note that μ is nowhere tangential and that $\mathbf{1}$ is an eigenfunction of $\mathbb{A}$. Thus, $\ker(\mathbb{A}) = \mathbb{R}\cdot\mathbf{1}$ follows from [2, Thm.12.1] since $(-\Delta, \partial_\mu)$ is regular elliptic. To determine the range of $\mathbb{A}$, fix $p > n/(1-\delta)$ and suppose first that $\mathbb{A}u = (f,g)$, that is,

$$-\Delta u = f \quad \text{in } \Omega^{\rm i}_\rho\,, \qquad \partial_\mu u = g \quad \text{on } \Gamma_\rho\ . \tag{3.13}$$

Since $\partial_\tau w = (\vec{z}\times\nabla w)\cdot\nu$ and $\mathrm{div}(\vec{z}\times\nabla w) = 0$, we have by Gauss' Theorem

$$\int_{\Gamma_\rho} \partial_\tau w\,{\rm d}\sigma = \int_{\Omega^{\rm i}_\rho} \mathrm{div}(\vec{z}\times\nabla w)\,{\rm d}x = 0\,, \quad w \in W^2_p(\Omega^{\rm i}_\rho)\ . \tag{3.14}$$

Therefore,

$$\int_{\Omega^{\rm i}_\rho} f\,{\rm d}x = \frac{|\Theta_{\rm i}|^2}{\alpha_{\rm i}}\int_{\Gamma_\rho}\left(-\frac{\alpha_{\rm i}}{|\Theta_{\rm i}|^2}\partial_\nu u - \frac{\beta_{\rm i}}{|\Theta_{\rm i}|^2}\partial_\tau u\right){\rm d}\sigma = \frac{|\Theta_{\rm i}|^2}{\alpha_{\rm i}}\int_{\Gamma_\rho} g\,{\rm d}\sigma\ .$$

For the reverse inclusion we use a Fredholm argument. By [2, Lem.5.1] there is a coretraction

$$M \in \mathcal{L}\big(W^{1-1/p}_p(\Gamma_\rho), W^2_p(\Omega^{\rm i}_\rho)\big)$$

for the boundary operator such that $\partial_\mu Mg = g$ for $g \in W^{1-1/p}_p(\Gamma_\rho)$. Then

$$R := \Delta M \in \mathcal{L}\big(W^{1-1/p}_p(\Gamma_\rho), L_p(\Omega^{\rm i}_\rho)\big)\,,$$

and finding a solution $u \in W^2_p(\Omega^{\rm i}_\rho)$ to problem (3.13) for a given $(f,g) \in L_p(\Omega^{\rm i}_\rho) \times W^{1-1/p}_p(\Gamma_\rho)$ is equivalent to finding a solution $v \in W^2_p(\Omega^{\rm i}_\rho)$ to

$$-\Delta v = f + Rg \quad \text{in } \Omega^{\rm i}_\rho\,, \qquad \partial_\mu v = 0 \quad \text{on } \Gamma_\rho\,, \tag{3.15}$$

and setting $u := v + Mg$. Let

$$W^2_{p,\mu}(\Omega^{\rm i}_\rho) := \big\{v \in W^2_p(\Omega^{\rm i}_\rho)\,;\ \partial_\mu v = 0 \text{ on } \Gamma_\rho\big\}$$

and consider the closed linear operator T on $L_p(\Omega^{\rm i}_\rho)$ given by $Tv := -\Delta v$ for $v \in W^2_{p,\mu}(\Omega^{\rm i}_\rho)$. Then, as above, $\ker(T) = \mathbb{R}\cdot\mathbf{1}$, and T has compact resolvent. Hence T is a Fredholm operator on $L_p(\Omega^{\rm i}_\rho)$ with index zero. Since obviously $W^2_{p,\mu}(\Omega^{\rm i}_\rho)$ is dense in $L_p(\Omega^{\rm i}_\rho)$, we also obtain for its dual T' that $\ker(T') = \mathbb{R}\cdot f'$ for some $f' \in L_{p'}(\Omega^{\rm i}_\rho)$, where p' denotes the dual exponent of p. Moreover, $Tv = h$ with $h \in L_p(\Omega^{\rm i}_\rho)$ is solvable for $v \in W^2_{p,\mu}(\Omega^{\rm i}_\rho)$ if and only if $\langle f', h\rangle = 0$. However, since (3.14) ensures

$$\int_{\Omega^{\rm i}_\rho} \Delta v\ {\rm d}x = -\frac{|\Theta_{\rm i}|^2}{\alpha_{\rm i}}\int_{\Gamma_\rho} \partial_\mu v\ {\rm d}\sigma = 0\,, \quad v \in W^2_{p,\mu}(\Omega^{\rm i}_\rho)\,,$$

we may take $f' = \mathbf{1}$, that is, $\ker(T') = \mathbb{R}\cdot\mathbf{1}$, and $Tv = f + Rg$ is solvable for $v \in W^2_{p,\mu}(\Omega^{\rm i}_\rho)$ with

$$(f,g) \in L_p(\Omega^{\rm i}_\rho) \times W^{1-1/p}_p(\Gamma_\rho)$$

if and only if

$$0 = \int_{\Omega_\rho^{\mathrm{i}}} (f + Rg)\ \mathrm{d}x = \int_{\Omega_\rho^{\mathrm{i}}} f\ \mathrm{d}x - \frac{|\Theta_{\mathrm{i}}|^2}{\alpha_{\mathrm{i}}} \int_{\Gamma_\rho} \partial_\mu g\ \mathrm{d}\sigma\,, \tag{3.16}$$

the last equality being again due to (3.14). Finally, if u is the solution to (3.13) for a given

$$(f, g) \in h^\delta(\Omega_\rho^{\mathrm{i}}) \times h^{1+\delta}(\Gamma_\rho) \subset L_p(\Omega_\rho^{\mathrm{i}}) \times W_p^{1-1/p}(\Gamma_\rho)$$

satisfying (3.16), then $u \in W_p^2(\Omega_\rho^{\mathrm{i}}) \hookrightarrow C^{1+\delta}(\overline{\Omega_\rho^{\mathrm{i}}})$ by the choice of p. Using Schauder regularity theory [12, Thm.6.15] for

$$-\Delta \bar{u} = f \quad \text{in } \Omega_\rho^{\mathrm{i}}\,, \qquad \bar{u} + \partial_\mu \bar{u} = g + u \quad \text{on } \Gamma_\rho\,,$$

then easily gives $u = \bar{u} \in h^{2+\delta}(\Omega_\rho^{\mathrm{i}})$. This proves the claimed characterization of $\mathrm{im}(\mathbb{A})$. □

Proof of Proposition 3.4. For $\rho \in \mathcal{V}$ fixed, set

$$P(\rho)h := \frac{|\phi_\rho^*(\nabla N_\rho)|}{|\Gamma_\rho|} \int_{\Gamma_\rho} \frac{\phi_*^\rho h}{|\nabla N_\rho|}\, \mathrm{d}\sigma\,, \quad h \in h^{1+\delta}(\mathbb{S}^1)\,,$$

and define an operator $\mathbb{A}_{\mathrm{i}}(\rho) : h^{2+\delta}(B^{\mathrm{i}}) \to h^\delta(B^{\mathrm{i}}) \times h^{1+\delta}(\mathbb{S}^1)$ by

$$\mathbb{A}_{\mathrm{i}}(\rho)u := \big(D(\rho)\mathcal{A}_{\mathrm{i}}(\rho)u\,,\, d(\rho)\mathcal{B}_{\mathrm{i}}(\rho)u\big), \quad u \in h^{2+\delta}(B^{\mathrm{i}})\,,$$

where $D(\rho) := |\det \partial_x \phi_\rho|$ and $d(\rho) := 1 + \rho$. Then, since $D(\rho) > 0$ and $d(\rho) > 0$, we readily obtain from Lemma 3.5 that $\ker\big(\mathbb{A}_{\mathrm{i}}(\rho)\big) = \mathbb{R}\cdot\mathbf{1}$. Moreover, $(f, g) \in \mathrm{im}\big(\mathbb{A}_{\mathrm{i}}(\rho)\big)$ if and only if $(\hat{f}, \hat{g}) \in \mathrm{im}(\mathbb{A})$, where

$$\hat{f} := \phi_*^\rho \frac{f}{D(\rho)} \in h^\delta(\Omega_\rho^{\mathrm{i}})\,, \qquad \hat{g} := \frac{1}{|\nabla N_\rho|} \phi_*^\rho \frac{g}{d(\rho)} \in h^{1+\delta}(\Gamma_\rho)\ .$$

By virtue of (3.5) we have that

$$|\nabla N_\rho| = \frac{\sqrt{\dot{\rho} + (1+\rho)^2}}{1+\rho} \quad \text{on} \quad \mathbb{S}^1\,, \tag{3.17}$$

and we infer from Lemma 3.5 that

$$\mathrm{im}\big(\mathbb{A}_{\mathrm{i}}(\rho)\big) = \left\{(f, g) \in h^\delta(B^{\mathrm{i}}) \times h^{1+\delta}(\mathbb{S}^1)\,;\, \int_{B^{\mathrm{i}}} f\, \mathrm{d}x = \frac{|\Theta_{\mathrm{i}}|^2}{\alpha_{\mathrm{i}}} \int_{\mathbb{S}^1} g\, \mathrm{d}\sigma\right\} =: Y$$

which is independent of ρ. Consequently, the map

$$\mathbb{A}_{\mathrm{i}}(\rho) : h^{2+\delta}(B^{\mathrm{i}})\big/_{\mathbb{R}\cdot\mathbf{1}} \longrightarrow Y$$

is an isomorphism. Furthermore, $v = \mathcal{N}(\rho)(f, g) := \mathbb{A}_{\mathrm{i}}(\rho)^{-1}(f, g)$ is a unique solution in $h^{2+\delta}(B^{\mathrm{i}})\big/_{\mathbb{R}\cdot\mathbf{1}}$ to

$$D(\rho)\mathcal{A}_{\mathrm{i}}(\rho)v = f \quad \text{on } B^{\mathrm{i}}\,, \qquad d(\rho)\mathcal{B}_{\mathrm{i}}(\rho)v = g \quad \text{on } \mathbb{S}^1$$

for each $\rho \in \mathcal{V}$ and $(f, g) \in Y$. As $D(\rho)$ and $d(\rho)$ are analytic in ρ, we deduce that

$$\mathcal{N} : \mathcal{V} \to \mathcal{L}\left(Y, h^{2+\delta}(B^{\mathrm{i}})\big/_{\mathbb{R}\cdot\mathbf{1}}\right)$$

is analytic. Now, we easily check with the help of (3.17) that

$$\int_{\mathbb{S}^1} d(\rho)\big(1 - P(\rho)\big)h\, \mathrm{d}\sigma = 0\,,$$

whence $\big(0, d(\rho)(1 - P(\rho))h\big) \in Y$ for $h \in h^{1+\delta}(\mathbb{S}^1)$. Thus $u := \mathcal{N}(\rho)\big(0, d(\rho)(1 - P(\rho))h\big)$ is the unique solution in $h^{2+\delta}(B^{\mathrm{i}})\big/_{\mathbb{R}\cdot\mathbf{1}}$ to

$$\mathcal{A}_{\mathrm{i}}(\rho)u = 0 \quad \text{on } B^{\mathrm{i}}\,, \qquad \mathcal{B}_{\mathrm{i}}(\rho)u = h - P(\rho)h \quad \text{on } \mathbb{S}^1\ .$$

By (3.17) we have

$$P(\rho)h = \frac{\sqrt{\dot\rho + (1+\rho)^2}}{(1+\rho)\int_0^{2\pi}\sqrt{\dot\rho+(1+\rho)^2}\,\mathrm{d}\theta}\int_0^{2\pi} h(\theta)\,\big(1+\rho(\theta)\big)\,\mathrm{d}\theta\,,$$

hence $P(\rho)$ depends analytically on ρ and so does $u = u(\rho)$. Given $\rho \in \mathcal{V}$ and $h \in h^{1+\delta}(\mathbb{S}^1)$, problem (3.6)-(3.8) thus admits a unique solution $Q_{\mathrm{i}} = S(\rho, h)$ and

$$[\rho \mapsto S(\rho,\cdot)] : \mathcal{V} \to \mathcal{L}\big(h^{1+\delta}(\mathbb{S}^1), h^{2+\delta}(B^{\mathrm{i}})\big)$$

is analytic. This proves Proposition 3.4. □

Proposition 3.6. *Given $\rho \in \mathcal{V}$ and $g \in h^{2+\delta}(\mathbb{S}^1)$, there is a unique solution $Q_{\mathrm{o}} = T(\rho, g) \in h^{2+\delta}(B^{\mathrm{o}})$ to (3.9)-(3.10). Moreover, the map $[\rho \mapsto T(\rho,\cdot)] : \mathcal{V} \to \mathcal{L}\big(h^{2+\delta}(\mathbb{S}^1), h^{2+\delta}(B^{\mathrm{o}})\big)$ is analytic.*

Proof. Let $\rho \in \mathcal{V}$ and define an operator $\mathbb{A}_{\mathrm{o}}(\rho) : h^{2+\delta}(B^{\mathrm{o}}) \to h^{\delta}(B^{\mathrm{o}}) \times h^{2+\delta}(\mathbb{S}^1) \times h^{1+\delta}(R\,\mathbb{S}^1)$ by

$$\mathbb{A}_{\mathrm{o}}(\rho)u := \big(\mathcal{A}_{\mathrm{o}}(\rho)u\,,\, u|_{\mathbb{S}^1}\,,\, \mathcal{B}_{\mathrm{o}}u\big), \quad u \in h^{2+\delta}(B^{\mathrm{o}})\ .$$

Then $\mathbb{A}_{\mathrm{o}}(\rho)$ is invertible. Indeed, to check its injectivity, let $\mathbb{A}_{\mathrm{o}}(\rho)u = 0$. Then, for $\bar u := \phi_*^\rho u$,

$$\begin{aligned} \Delta\bar u &= 0 && \text{in} \quad \Omega_\rho^{\mathrm{o}}\,,\\ \bar u &= 0 && \text{on} \quad \Gamma_\rho\,,\\ \alpha_{\mathrm{o}}\partial_\nu \bar u + \beta_{\mathrm{o}}\partial_\tau \bar u &= 0 && \text{on} \quad R\,\mathbb{S}^1\,, \end{aligned}$$

and Green's formula yields

$$\begin{aligned} 0 &= \int_{\Omega_\rho^{\mathrm{o}}} \bar u\Delta\bar u\,\mathrm{d}x = -\int_{\Omega_\rho^{\mathrm{o}}} |\nabla\bar u|^2\,\mathrm{d}x + \int_{\Gamma_\rho\cup R\mathbb{S}^1} \bar u\partial_\nu\bar u\,\mathrm{d}\sigma = -\int_{\Omega_\rho^{\mathrm{o}}}|\nabla\bar u|^2\,\mathrm{d}x - \frac{\beta_{\mathrm{o}}}{\alpha_{\mathrm{o}}}\int_{\Gamma_\rho\cup R\mathbb{S}^1}\frac12\partial_\tau\bar u^2\,\mathrm{d}\sigma\\ &= -\int_{\Omega_\rho^{\mathrm{o}}}|\nabla\bar u|^2\,\mathrm{d}x - \frac{\beta_{\mathrm{o}}}{2\alpha_{\mathrm{o}}}\int_{\Omega_\rho^{\mathrm{o}}} \operatorname{div}\begin{pmatrix}-\partial_2\bar u^2\\ \partial_1\bar u^2\end{pmatrix}\mathrm{d}x = -\int_{\Omega_\rho^{\mathrm{o}}}|\nabla\bar u|^2\,\mathrm{d}x\ . \end{aligned}$$

This entails $\bar u = 0$, i.e. $u = 0$. To determine the range of $\mathbb{A}_{\mathrm{o}}(\rho)$ consider again the transformed problem

$$\begin{aligned} \Delta\bar u &= \bar f && \text{in} \quad \Omega_\rho^{\mathrm{o}}\,, && (3.18)\\ \bar u &= \bar g && \text{on} \quad \Gamma_\rho\,, && (3.19)\\ \alpha_{\mathrm{o}}\partial_\nu \bar u + \beta_{\mathrm{o}}\partial_\tau \bar u &= \bar h && \text{on} \quad R\,\mathbb{S}^1\ . && (3.20) \end{aligned}$$

By means of a coretraction as in the proof of Lemma 3.5 we may assume $\bar g = 0$ and $\bar h = 0$. However, by [2],

$$\lambda - \Delta : \big\{v \in W_2^2(\Omega_\rho^{\mathrm{o}})\,;\, v = 0 \text{ on } \Gamma_\rho\,,\, (\alpha_{\mathrm{o}}\partial_\nu + \beta_{\mathrm{o}}\partial_\tau)v = 0 \text{ on } R\,\mathbb{S}^1\big\} \longrightarrow L_2(\Omega_\rho^{\mathrm{o}})$$

is invertible for $\lambda \geq 0$ sufficiently large, and the same argument as used to prove its injectivity implies that we actually may take $\lambda = 0$. Thus, (3.18)-(3.20) is uniquely solvable for each $\bar f \in L_2(\Omega_\rho^{\mathrm{o}})$, $\bar g \in H^{3/2}(\Gamma_\rho)$, and $\bar h \in H^{1/2}(R\,\mathbb{S}^1)$. Schauder regularity theory ensures $\bar u \in h^{2+\delta}(\Omega_\rho^{\mathrm{o}})$ provided $\bar f \in h^{\delta}(\Omega_\rho^{\mathrm{o}})$, $\bar g \in h^{2+\delta}(\Gamma_\rho)$, and $\bar h \in h^{1+\delta}(R\,\mathbb{S}^1)$. Consequently, $\mathbb{A}_{\mathrm{o}}(\rho)$ is surjective and thus invertible, and the claim on the analyticity of $T(\rho,\cdot) = \mathbb{A}_{\mathrm{o}}(\rho)^{-1}(0,\cdot,0)$ follows from Lemma 3.2. □

Next, we state a multiplier result that we shall use later. The proof is a straightforward modification of the case $r = s$ in [3].

Proposition 3.7. *Let $r, s \in (0,\infty)\setminus\mathbb{N}$ and $(M_n)_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$ satisfying*

(i) $\sup_{n\in\mathbb{Z}\setminus\{0\}} |n|^{r-s}|M_n| < \infty\,,$

(ii) $\sup_{n\in\mathbb{Z}\setminus\{0\}} |n|^{r-s+1}|M_{n+1}-M_n| < \infty$,
(iii) $\sup_{n\in\mathbb{Z}\setminus\{0\}} |n|^{r-s+2}|M_{n+2}-2M_{n+1}+M_n| < \infty$.

Then the mapping

$$\sum_{n\in\mathbb{Z}} \hat{h}_n e^{in\theta} \longmapsto \sum_{n\in\mathbb{Z}} \hat{h}_n M_n e^{in\theta}$$

belongs to $\mathcal{L}\big(C^s(\mathbb{S}^1), C^r(\mathbb{S}^1)\big)$.

We now focus on the evolution equation (3.12), which we may rewrite, using linearity of the solution operators S and T, as

$$\big(1-\mathcal{R}(\rho)\big)\partial_t\rho = \mathcal{B}_{\mathrm{o}}(\rho)T\big(\rho, -\mathcal{K}(\rho)\big) \tag{3.21}$$

with

$$\mathcal{R}(\rho)z := \mathcal{B}_{\mathrm{o}}(\rho)T\big(\rho, S(\rho,z)\big)\,, \quad z\in h^{1+\delta}(\mathbb{S}^1)\,, \quad \rho\in\mathcal{V}\,.$$

To solve (3.21) for $\partial_t\rho$, we use the following

Lemma 3.8. *There is an open zero neighborhood* $\mathcal{W}\subset\mathcal{V}$ *in* $h^{4+\delta}(\mathbb{S}^1)$ *such that* $1-\mathcal{R}(\rho)$ *is an isomorphism on* $h^{1+\delta}(\mathbb{S}^1)$ *for each* $\rho\in\mathcal{W}$.

Proof. By the smooth dependence on ρ stated in Lemma 3.3, Proposition 3.4, and Proposition 3.6, it suffices to prove that $1-\mathcal{R}(0)\in\mathcal{L}\big(h^{1+\delta}(\mathbb{S}^1)\big)$ is invertible. To do this we derive its Fourier expansion in polar coordinates (r,θ). To compute $Q_{\mathrm{i}} = S(0,h)$ we first note that, for $\rho=0$, problem (3.6)-(3.8) becomes

$$\begin{aligned}
\Big(\frac{1}{r}\partial_r(r\partial_r)+\frac{1}{r^2}\partial_\theta^2\Big)Q_{\mathrm{i}} &= 0 && \text{in} \quad [r<1]\,,\\
\frac{-1}{|\Theta_{\mathrm{i}}|}\left(\alpha_{\mathrm{i}}\partial_r-\beta_{\mathrm{i}}\partial_\theta\right)Q_{\mathrm{i}} &= h-\frac{1}{2\pi}\int_0^{2\pi}h(\theta)\,\mathrm{d}\theta && \text{on} \quad [r=1]\,,\\
\int_0^{2\pi}Q_{\mathrm{i}}(1,\theta)\,\mathrm{d}\theta &= \int_0^{2\pi}h(\theta)\,\mathrm{d}\theta\,,
\end{aligned}$$

which, for a given $h\in h^{1+\delta}(\mathbb{S}^1)$ with expansion

$$h(\theta)=\sum_{n\in\mathbb{Z}}\hat{h}_n e^{in\theta}\,, \tag{3.22}$$

has the unique solution

$$Q_{\mathrm{i}}(r,\theta)=\hat{h}_0+\sum_{n\in\mathbb{Z}\setminus\{0\}}\hat{h}_n\frac{|\Theta_{\mathrm{i}}|^2}{-|n|\alpha_{\mathrm{i}}+in\beta_{\mathrm{i}}}r^{|n|}e^{in\theta}\,. \tag{3.23}$$

Similarly, for $\rho=0$, problem (3.9)-(3.10) reads as

$$\begin{aligned}
\Big(\frac{1}{r}\partial_r(r\partial_r)+\frac{1}{r^2}\partial_\theta^2\Big)Q_{\mathrm{o}} &= 0 && \text{in} \quad [1<r<R]\,,\\
Q_{\mathrm{o}} &= g && \text{on} \quad [r=1]\,,\\
\Big(\alpha_{\mathrm{o}}\partial_r-\frac{\beta_{\mathrm{o}}}{R}\partial_\theta\Big)Q_{\mathrm{o}} &= 0 && \text{on} \quad [r=R]\,,
\end{aligned}$$

and for $g\in h^{2+\delta}(\mathbb{S}^1)$ with expansion

$$g(\theta)=\sum_{n\in\mathbb{Z}}\hat{g}_n e^{in\theta}\,,$$

the unique solution $Q_{\rm o} = T(0,g)$ is given by

$$Q_{\rm o}(r,\theta) = \hat{g}_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \hat{g}_n \left(\frac{\Theta_{\rm o}}{\Theta_{\rm o} + R^{2n}\bar{\Theta}_{\rm o}} r^n + \frac{\bar{\Theta}_{\rm o}}{\bar{\Theta}_{\rm o} + R^{-2n}\Theta_{\rm o}} r^{-n} \right) e^{in\theta} \; . \tag{3.24}$$

Therefore, given $h \in h^{1+\delta}(\mathbb{S}^1)$ with expansion (3.22), we have

$$\mathcal{R}(0)h = \frac{-1}{|\Theta_{\rm o}|} \left(\alpha_{\rm o}\partial_r - \beta_{\rm o}\partial_\theta\right) T\big(0, S(0,h)\big) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \hat{h}_n \, l_n e^{in\theta} \,,$$

where

$$l_n := \frac{\big(1 - R^{2n}\big)|\Theta_{\rm i}|^2}{\big(\mathrm{sign}(n)\alpha_{\rm i} - i\beta_{\rm i}\big)\big(\Theta_{\rm o} + R^{2n}\bar{\Theta}_{\rm o}\big)} \,, \quad n \in \mathbb{Z}\setminus\{0\} \; . \tag{3.25}$$

We next use Proposition 3.7 with $M_n := (1-l_n)^{-1}$ to check that $1 - \mathcal{R}(0)$ is invertible on $C^{1+\delta}(\mathbb{S}^1)$. Note that $l_n \neq 1$ for each $n \in \mathbb{Z}\setminus\{0\}$ since $\alpha_j > 0$, $j = {\rm i}, {\rm o}$. Also,

$$\lim_{n\to\infty} l_n = -\frac{\Theta_{\rm i}\Theta_{\rm o}}{|\Theta_{\rm o}|^2} \neq 1 \,, \qquad \lim_{n\to-\infty} l_n = -\frac{\bar{\Theta}_{\rm i}\bar{\Theta}_{\rm o}}{|\Theta_{\rm o}|^2} \neq 1 \tag{3.26}$$

so that

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} |M_n| < \infty \; . \tag{3.27}$$

Next,

$$M_{n+1} - M_n = \frac{l_{n+1} - l_n}{(1-l_{n+1})(1-l_n)} \,,$$

where, for $n \geq 1$,

$$l_{n+1} - l_n = \frac{O(R^{2n+2})}{\bar{\Theta}_{\rm o}^2 R^{4n+2} + O(R^{2n+2})} \,, \quad n \geq 1 \,, \qquad l_{n+1} - l_n = \frac{O(R^{2n})}{\bar{\Theta}_{\rm o}^2 + O(R^{2n})} \,, \quad n \leq -1 \; .$$

Since $nR^{-2n} \to 0$ as $n \to \infty$ and $nR^{2n} \to 0$ as $n \to -\infty$, it follows from (3.27) that

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} |n| \, |M_{n+1} - M_n| < \infty \; .$$

Finally, as above, we obtain from (3.26) and (3.27) that

$$\begin{aligned} n^2 \left| M_{n+2} - 2M_{n+1} + M_n \right| &= n^2 \left| \frac{l_{n+2} - 2l_{n+1} + l_n + l_n(l_{n+1} - l_{n+2}) + l_{n+2}(l_{n+1} - l_n)}{(1-l_{n+2})(1-l_{n+1})(1-l_n)} \right| \\ &\leq (\sup|M_n|)^3 \, \left[(1 + |l_n|)\, n^2 \left| l_{n+2} - l_{n+1} \right| + (1 + |l_{n+2}|)\, n^2 \left| l_{n+1} - l_n \right| \right] \\ &\leq c \sup_n \left(n^2 \left| l_{n+1} - l_n \right| \right) < \infty \; . \end{aligned}$$

Consequently, Lemma 3.7 implies $\big(1 - \mathcal{R}(0)\big)^{-1} \in \mathcal{L}\big(C^{1+\delta}(\mathbb{S}^1)\big)$. But then $\big(1 - \mathcal{R}(0)\big)^{-1}$ is also a bounded operator on

$$H^s(\mathbb{S}^1) = \left\{ h \in L_2(\mathbb{S}^1) \,;\, \|h\|_{H^s} := \sum_{n\in\mathbb{Z}} (1+n^2)^s |\hat{h}_n|^2 < \infty \right\}$$

for each $s > 0$ due to (3.27). Recalling that $H^s(\mathbb{S}^1)$ is densely embedded in $h^{1+\delta}(\mathbb{S}^1)$ provided $s > 5/2$, we deduce that $\big(1 - \mathcal{R}(0)\big)^{-1} \in \mathcal{L}\big(h^{1+\delta}(\mathbb{S}^1)\big)$. □

According to Lemma 3.8 and (3.21), we are led to look for solutions

$$\rho \in C\big([0,T], \mathcal{W}\big) \cap C^1\big([0,T], h^{1+\delta}(\mathbb{S}^1)\big)$$

to the fully nonlinear equation

$$\partial_t \rho = \big(1 - \mathcal{R}(\rho)\big)^{-1} \mathcal{B}_{\rm o}(\rho) T\big(\rho \,, -\mathcal{K}(\rho)\big) =: F(\rho) \,, \quad t \in (0,T] \; . \tag{3.28}$$

The following proposition is instrumental in the computation of the linearization in zero of this evolution equation.

Proposition 3.9. *$F\in C^\infty\big(\mathcal{W},h^{1+\delta}(\mathbb{S}^1)\big)$ and, for $h\in h^{4+\delta}(\mathbb{S}^1)$,*

$$\partial F(0)[h]=\big(1-\mathcal{R}(0)\big)^{-1}\mathcal{B}_{\rm o}(0)T\big(0\,,\,(2(\gamma_{\rm i}-\gamma_{\rm o})+\sigma)h+\sigma\ddot{h}\big)\ .$$

In particular, if $h\in h^{4+\delta}(\mathbb{S}^1)$ with $h(\theta)=\sum_{n\in\mathbb{Z}}\hat{h}_n e^{in\theta}$, then

$$\partial F(0)[h](\theta)=\sum_{n\in\mathbb{Z}\setminus\{0\}}\hat{h}_n q_n e^{in\theta}$$

where, for $n\in\mathbb{Z}\setminus\{0\}$,

$$q_n:=\frac{A_n+i\,\mathrm{sign}(n)B}{A_n^2+B^2}\mu(n) \tag{3.29}$$

with

$$A_n:=\mathrm{sign}(n)\frac{R^{2n}+1}{R^{2n}-1}\alpha_{\rm o}+\alpha_{\rm i}\,,\quad B:=\beta_{\rm o}-\beta_{\rm i}\,,\quad \mu(n):=|n|\big(\sigma-2(\gamma_{\rm o}-\gamma_{\rm i})-\sigma n^2\big)\ .$$

Proof. Smoothness of the map F follows from Lemma 3.3, Proposition 3.4, Proposition 3.6, and Lemma 3.8. Let $h\in h^{4+\delta}(\mathbb{S}^1)$. Then

$$\begin{aligned}\partial F(0)[h]&=\partial\big(1-\mathcal{R}(\cdot)\big)^{-1}(0)[h]\,\mathcal{B}_{\rm o}(0)\,T\big(0\,,\,-\mathcal{K}(0)\big)+\big(1-\mathcal{R}(0)\big)^{-1}\,\partial\mathcal{B}_{\rm o}(0)[h]\,T\big(0\,,\,-\mathcal{K}(0)\big)\\&\quad+\big(1-\mathcal{R}(0)\big)^{-1}\,\mathcal{B}_{\rm o}(0)\,\partial_\rho T\big(0\,,\,-\mathcal{K}(0)\big)[h]+\big(1-\mathcal{R}(0)\big)^{-1}\,\mathcal{B}_{\rm o}(0)\,T\big(0\,,\,-\partial\mathcal{K}(0)[h]\big)\ .\end{aligned}$$

But $-\mathcal{K}(0)=-\sigma-\gamma_{\rm o}+\gamma_{\rm i}=:c\in\mathbb{R}$ and $T(0,c)=c$ by uniqueness, so $\partial\mathcal{B}_{\rm o}(0)[h]=0$ as this is the derivative of $\big(\rho\mapsto\mathcal{B}_{\rm o}(\rho)c=0\big)$ and similarly $\partial_\rho T\big(0\,,\,-\mathcal{K}(0)\big)[h]=0$ as this is the derivative of $\big(\rho\mapsto T(\rho,c)=c\big)$. The formula for $\partial F(0)[h]$ now follows from Lemma 3.3. To compute its Fourier expansion, consider $h\in h^{4+\delta}(\mathbb{S}^1)$ with $h(\theta)=\sum_{n\in\mathbb{Z}}\hat{h}_n e^{in\theta}$. Invoking (3.24) and recalling that $\mathcal{B}_{\rm o}(0)=-\big(\alpha_{\rm o}\partial_r-\beta_{\rm o}\partial_\theta\big)/|\Theta_{\rm o}|^2$ on $\mathbb{S}^1$, we obtain

$$\begin{aligned}\partial F(0)[h](\theta)&=\big(1-\mathcal{R}(0)\big)^{-1}\mathcal{B}_{\rm o}(0)T\big(0\,,\,(2(\gamma_{\rm i}-\gamma_{\rm o})+\sigma)h+\sigma\ddot{h}\big)(\theta)\\&=\sum_{n\in\mathbb{Z}\setminus\{0\}}\hat{h}_n\,\frac{n\big(2(\gamma_{\rm i}-\gamma_{\rm o})+\sigma-\sigma n^2\big)(R^{2n}-1)}{\Theta_{\rm o}+R^{2n}\bar{\Theta}_{\rm o}}\,\frac{1}{1-l_n}\,e^{in\theta}\end{aligned}$$

with l_n given by (3.25). Elementary calculations now lead to the assertion. □

Observe that $A_n=A_{-n}$ for $n\in\mathbb{Z}\setminus\{0\}$ and that

$$A_n\searrow\alpha_{\rm o}+\alpha_{\rm i}\quad\text{as}\quad|n|\to\infty\ . \tag{3.30}$$

The next proposition is a consequence of the previous lemma and fundamental for our well-posedness result.

Proposition 3.10. *$-\partial F(0)\in\mathcal{H}\big(h^{4+\delta}(\mathbb{S}^1),h^{1+\delta}(\mathbb{S}^1)\big)$, that is, $\partial F(0)\in\mathcal{L}\big(h^{4+\delta}(\mathbb{S}^1),h^{1+\delta}(\mathbb{S}^1)\big)$ is the generator of an analytic semigroup on $h^{1+\delta}(\mathbb{S}^1)$.*

Proof. Based on Proposition 3.7, we prove in a first step that $\big(\lambda-\partial F(0)\big)^{-1}\in\mathcal{L}\big(h^{1+\delta}(\mathbb{S}^1),h^{4+\delta}(\mathbb{S}^1)\big)$, where, according to Proposition 3.9 (with $q_0:=0$), we have that

$$\big(\lambda-\partial F(0)\big)^{-1}h=\sum_{n\in\mathbb{Z}}\hat{h}_n\,\frac{1}{\lambda-q_n}\,e^{in\theta}\,,\quad\mathrm{Re}\,\lambda\geq\lambda_*\,,$$

for $h(\theta)=\sum_{n\in\mathbb{Z}}\hat{h}_n e^{in\theta}$ and, by (3.30),

$$\lambda_*:=1+\frac{2|\gamma_{\rm o}-\gamma_{\rm i}|}{\alpha_{\rm o}+\alpha_{\rm i}}>\mathrm{Re}\,q_n\,,\quad n\in\mathbb{Z}\ .$$

Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \lambda_*$ be fixed and set $Q_n^\lambda := (\lambda - q_n)^{-1}$. Then

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} |n|^3 \left|Q_n^\lambda\right| < \infty\,, \tag{3.31}$$

since

$$\frac{q_n}{n^3} \longrightarrow \pm\frac{\alpha_{\rm o}+\alpha_{\rm i} \pm iB}{(\alpha_{\rm o}+\alpha_{\rm i})^2+B^2}\,\sigma \quad \text{as} \quad n \longrightarrow \pm\infty\ .$$

Next observe that, for each $m \in \mathbb{N}$,

$$|n|^m |A_{n+1} - A_n| = |n|^m \frac{2R^{2n}(R-1)}{(R^{2n}-1)(R^{2n+1}-1)} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \pm\infty\ .$$

Thus, letting $z_n := q_n/\mu(n)$ we derive from (3.30)

$$|n|^m |z_{n+1} - z_n| \leq \frac{A_{n+1}A_n}{(A_{n+1}^2+B^2)(A_n^2+B^2)}\, |n|^m\, |A_{n+1}-A_n| + |n|^m\, |A_{n+1}-A_n|\, B^2\,,$$

so, for each $m \in \mathbb{N}$,

$$|n|^m |z_{n+1} - z_n| \longrightarrow 0 \quad \text{as} \quad |n| \longrightarrow \infty\ . \tag{3.32}$$

Thus, (3.31) gives

$$\begin{aligned}|n|^4\, \left|Q_{n+1}^\lambda - Q_n^\lambda\right| &= \left|n^3 Q_{n+1}^\lambda\right|\, \left|n^3 Q_n^\lambda\right|\, \left|\frac{q_{n+1}-q_n}{n^2}\right| \\ &\leq c\frac{|z_{n+1}|\,|\mu(n+1)-\mu(n)|}{n^2} + c|z_{n+1}-z_n|\,|n|\, \left|\frac{\mu(n)}{n^3}\right|\ .\end{aligned}$$

Taking into account that (z_n) is bounded and that $|\mu(n)/n^3| \to \sigma$ as $|n| \to \infty$, we conclude

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} |n|^4 \left|Q_{n+1}^\lambda - Q_n^\lambda\right| < \infty\ . \tag{3.33}$$

In particular, we have shown

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} n^{-2}\, |q_{n+1} - q_n| < \infty\ . \tag{3.34}$$

Observe then that

$$\mu(n)\big[\mu(n+1)-\mu(n+2)\big] + \mu(n+2)\big[\mu(n+1)-\mu(n)\big] = O(n^4)\,,$$

hence, due to (3.32),

$$\begin{aligned}&\left|\frac{1}{n^4}\Big[q_n\,(q_{n+1}-q_{n+2}) + q_{n+2}\,(q_{n+1}-q_n)\Big]\right| \\ &\qquad\leq \left|z_n\, z_{n+1}\, \frac{\mu(n)\big[\mu(n+1)-\mu(n+2)\big] + \mu(n+2)\big[\mu(n+1)-\mu(n)\big]}{n^4}\right| \\ &\qquad\quad + \left|z_n\, \frac{\mu(n)}{n^3}\, \frac{\mu(n+1)}{n^3}\, (z_{n+1}-z_{n+2})\, n^2\right| + \left|z_n\, \frac{\mu(n)}{n^3}\, \frac{\mu(n+2)}{n^3}\, (z_{n+1}-z_{n+2})\, n^2\right| \\ &\qquad\quad + \left|z_{n+1}\, \frac{\mu(n+2)}{n^3}\, \frac{\mu(n+1)}{n^3}\, (z_{n+2}-z_n)\, n^2\right| \\ &\qquad\leq c\ .\end{aligned} \tag{3.35}$$

Writing

$$\begin{aligned}&|n|^5\, \left|Q_{n+2}^\lambda - 2Q_{n+1}^\lambda + Q_n^\lambda\right| \\ &\quad = \left|n^3 Q_{n+2}^\lambda\right|\, \left|n^3 Q_{n+1}^\lambda\right|\, \left|n^3 Q_n^\lambda\right|\, \left|\frac{\lambda(q_{n+2}-2q_{n+1}+q_n)}{n^4} + \frac{q_n(q_{n+1}-q_{n+2}) + q_{n+2}(q_{n+1}-q_n)}{n^4}\right|\end{aligned}$$

we deduce from (3.31) and (3.34) that

$$\sup_{n\in\mathbb{Z}\setminus\{0\}} |n|^5 \left|Q_{n+2}^\lambda - 2Q_{n+1}^\lambda + Q_n^\lambda\right| < \infty \ . \tag{3.36}$$

Consequently, $\big(\lambda - \partial F(0)\big)^{-1} \in \mathcal{L}\big(C^{1+\delta}(\mathbb{S}^1), C^{4+\delta}(\mathbb{S}^1)\big)$ for $\mathrm{Re}\,\lambda \ge \lambda_*$ by Lemma 3.7 and (3.31), (3.33), and (3.36). Since (3.31) also ensures

$$\big(\lambda - \partial F(0)\big)^{-1} \in \mathcal{L}\big(H^s(\mathbb{S}^1)\big)\,, \quad s > 0\,, \tag{3.37}$$

we conclude $\big(\lambda - \partial F(0)\big)^{-1} \in \mathcal{L}\big(h^{1+\delta}(\mathbb{S}^1), h^{4+\delta}(\mathbb{S}^1)\big)$ for $\mathrm{Re}\,\lambda \ge \lambda_*$ as in the proof of Lemma 3.8.

The second step consists of proving the resolvent estimate

$$|\lambda| \left\|\big(\lambda - \partial F(0)\big)^{-1}\right\|_{\mathcal{L}(h^{1+\delta}(\mathbb{S}^1))} \le c\,, \quad \mathrm{Re}\,\lambda \ge \lambda_* \ . \tag{3.38}$$

Since $\mathrm{Re}\,q_n < 0$ for $|n|$ sufficiently large, elementary calculations show that

$$|\lambda - q_n|^2 \ge c_0|\lambda|^2\,, \quad \mathrm{Re}\,\lambda \ge \lambda_*\,, \quad n\in\mathbb{Z}\,, \tag{3.39}$$

for some $c_0 > 0$. Thus, setting $S_n^\lambda := \lambda(\lambda - q_n)^{-1}$it follows

$$\sup_{n\in\mathbb{Z}\setminus\{0\}\,,\,\mathrm{Re}\,\lambda\ge\lambda_*} |S_n^\lambda| < \infty \ . \tag{3.40}$$

Similarly, there is $c_1 > 0$ such that

$$|\lambda - q_n|^2 \ge c_1|q_n|^2\,, \quad \mathrm{Re}\,\lambda \ge \lambda_*\,, \quad n\in\mathbb{Z}\,, \tag{3.41}$$

and we thus obtain from

$$|n| \left|S_{n+1}^\lambda - S_n^\lambda\right| = \left|S_{n+1}^\lambda\right| \left|\frac{n^3}{\lambda - q_n}\right| \left|\frac{q_{n+1} - q_n}{n^2}\right|$$

together with (3.34), (3.40), and (3.41) combined with (3.31) that

$$\sup_{n\in\mathbb{Z}\setminus\{0\}\,,\,\mathrm{Re}\,\lambda\ge\lambda_*} |n| \left|S_{n+1}^\lambda - S_n^\lambda\right| < \infty \ . \tag{3.42}$$

Finally, noticing that the right hand side of

$$\begin{aligned}&\left|\frac{q_{n+2} - 2q_{n+1} + q_n}{n}\right| \\ &\qquad \le |z_{n+2}| \left|\frac{\mu(n+2) - 2\mu(n+1) + \mu(n)}{n}\right| + 2\left|\frac{\mu(n+1)}{n^3}\right| |z_{n+2} - z_{n+1}|\, n^2 + \left|\frac{\mu(n)}{n^3}\right| |z_n - z_{n+1}|\, n^2\end{aligned}$$

is bounded by (3.32) and writing

$$\begin{aligned}&n^2 \left|S_{n+2}^\lambda - 2S_{n+1}^\lambda + S_{n+2}^\lambda\right| \\ &\qquad = \left|S_{n+2}^\lambda\right| \left|n^3 Q_{n+1}^\lambda\right| \left|\frac{n^3}{\lambda - q_n}\right| \frac{1}{n^4} \left|\lambda(q_{n+2} - 2q_{n+1} + q_n) + q_n(q_{n+1} - q_{n+2}) + q_{n+2}(q_{n+1} - q_n)\right|\end{aligned}$$

we deduce from (3.31), (3.35), (3.39), and (3.40)

$$\sup_{n\in\mathbb{Z}\setminus\{0\}\,,\,\mathrm{Re}\,\lambda\ge\lambda_*} n^2 \left|S_{n+2}^\lambda - 2S_{n+1}^\lambda + S_n^\lambda\right| < \infty \ . \tag{3.43}$$

Therefore, Lemma 3.7 and (3.40), (3.42), and (3.43) imply

$$|\lambda| \left\|\big(\lambda - \partial F(0)\big)^{-1}\right\|_{\mathcal{L}(C^{1+\delta}(\mathbb{S}^1))} \le c\,, \quad \mathrm{Re}\,\lambda \ge \lambda_*\,,$$

whence (3.38) due to (3.37). This proves the assertion. □

Now we are in a position to establish a well-posedness result regarding equation (3.28).

Theorem 3.11. *There exists an open zero neighborhood $\mathcal{O} \subset \mathcal{V}$ in $h^{4+\delta}(\mathbb{S}^1)$ such that for each $\rho_0 \in \mathcal{O}$ there is $T := T(\rho_0) > 0$ and a unique solution*

$$\rho \in C\big([0,T], \mathcal{V}\big) \cap C^1\big([0,T], h^{1+\delta}(\mathbb{S}^1)\big)$$

to

$$\partial_t \rho = F(\rho)\,, \quad t > 0\,, \qquad \rho(0) = \rho_0\ .$$

Moreover, $\rho\big([0,T]\big) \subset \mathcal{O}$.

Proof. We shall invoke [14, Thm.8.1]. Fix $\xi \in (0,\delta)$ and put $\vartheta := (\delta - \xi)/3$. Set

$$E := h^{1+\xi}(\mathbb{S}^1)\,, \quad E_0 := h^{1+\delta}(\mathbb{S}^1)\,, \quad E_1 := h^{4+\delta}(\mathbb{S}^1)$$

in [14, Thm.8.1]. As $\delta \in (0,1)$ was arbitrary in Proposition 3.10, it follows that

$$-\partial F(0) \in \mathcal{H}\big(h^{4+\xi}(\mathbb{S}^1), h^{1+\xi}(\mathbb{S}^1)\big)\ .$$

Thus, since $\mathcal{H}\big(h^{4+\xi}(\mathbb{S}^1), h^{1+\xi}(\mathbb{S}^1)\big)$ is open in $\mathcal{L}\big(h^{4+\xi}(\mathbb{S}^1), h^{1+\xi}(\mathbb{S}^1)\big)$, there is an open zero neighborhood U_ξ in $h^{4+\xi}(\mathbb{S}^1)$ such that $-\partial F(\rho) \in \mathcal{H}\big(h^{4+\xi}(\mathbb{S}^1), h^{1+\xi}(\mathbb{S}^1)\big)$ for each $\rho \in U_\xi$. Then $\mathcal{O} := U_\xi \cap \mathcal{W}$ with $\mathcal{W}$ from Lemma 3.8 is an open zero neighborhood in $h^{4+\delta}(\mathbb{S}^1)$. Furthermore, $\partial F(\rho) : h^{4+\delta}(\mathbb{S}^1) \to h^{1+\delta}(\mathbb{S}^1)$ for $\rho \in \mathcal{O}$ is the part of $-\partial F(\rho) \in \mathcal{H}\big(h^{4+\xi}(\mathbb{S}^1), h^{1+\xi}(\mathbb{S}^1)\big)$ in

$$h^{1+\delta}(\mathbb{S}^1) \doteq \big(h^{1+\xi}(\mathbb{S}^1), h^{4+\xi}(\mathbb{S}^1)\big)^0_{\vartheta,\infty}$$

with continuous interpolation functor $(\cdot,\cdot)^0_{\vartheta,\infty}$ and

$$\big\{h \in h^{4+\xi}(\mathbb{S}^1)\,;\, \partial F(\rho)[h] \in h^{1+\delta}(\mathbb{S}^1)\big\} = h^{4+\delta}(\mathbb{S}^1)\ .$$

Now the assertion is a consequence of [14, Thm.8.1]. □

To finish off the proof of Theorem 2.1 let

$$\rho \in C\big([0,T], \mathcal{V}\big) \cap C^1\big([0,T], h^{1+\delta}(\mathbb{S}^1)\big)$$

be the solution to (3.28) for a given initial value $\rho_0 \in \mathcal{O}$. Then

$$Q_{\rm i} := S\big(\rho, \partial_t \rho\big) \in C\big([0,T], h^{2+\delta}(B^{\rm i})\big)\,, \qquad Q_{\rm o} := T\big(\rho, Q_{\rm i} - \mathcal{K}(\rho)\big) \in C\big([0,T], h^{2+\delta}(B^{\rm o})\big)$$

by Lemma 3.3, Proposition 3.4, and Proposition 3.6. Since ρ solves (3.12), it follows, for $\rho = \rho(t)$ with $t \in [0,T]$ fixed, that

$$\frac{\phi_*^\rho \partial_t \rho}{|\nabla N_\rho|} = \frac{-1}{|\Theta_{\rm o}|^2}\Big(\alpha_{\rm o}\partial_{\nu_\rho}\big(\phi_*^\rho Q_{\rm o}\big) + \beta_{\rm o}\partial_{\tau_\rho}\big(\phi_*^\rho Q_{\rm o}\big)\Big) \quad \text{on} \quad \Gamma_\rho\ .$$

Recalling from (3.9) that $\Delta\big(\phi_*^\rho Q_{\rm o}\big) = 0$ and $\operatorname{div}\big(\vec{z} \times \nabla\big(\phi_*^\rho Q_{\rm o}\big)\big) = 0$ in $\Omega_\rho^{\rm o}$, we deduce from Gauss' Theorem

$$\frac{-1}{|\Theta_{\rm o}|^2}\int_{\Gamma_\rho}\big(\alpha_{\rm o}\partial_{\nu_\rho}\big(\phi_*^\rho Q_{\rm o}\big) + \beta_{\rm o}\partial_{\tau_\rho}\big(\phi_*^\rho Q_{\rm o}\big)\big)\,\mathrm{d}\sigma = \frac{-1}{|\Theta_{\rm o}|^2}\int_{R\mathbb{S}^1}\big(\alpha_{\rm o}\partial_{\nu}\big(\phi_*^\rho Q_{\rm o}\big) + \beta_{\rm o}\partial_{\tau}\big(\phi_*^\rho Q_{\rm o}\big)\big)\,\mathrm{d}\sigma$$

and thus, due to $\phi_*^\rho Q_{\rm o} = Q_{\rm o}$ and (3.11) on $R\,\mathbb{S}^1$, that

$$\int_{\Gamma_\rho} \frac{\phi_*^\rho \partial_t \rho}{|\nabla N_\rho|}\,\mathrm{d}\sigma = 0\,, \quad t \in [0,T]\ . \tag{3.44}$$

Consequently, with

$$P_{\rm i}(t) := \phi_*^{\rho(t)} Q_{\rm i}(t) \in h^{2+\delta}(\Omega^{\rm i}_{\rho(t)})\,, \qquad P_{\rm o}(t) := \phi_*^{\rho(t)} Q_{\rm o}(t) h^{2+\delta}(\Omega^{\rm o}_{\rho(t)})$$

for $t \in [0,T]$ we obtain a solution $(\rho, P_{\rm i}, P_{\rm o})$ to (2.11)-(2.14) which is unique up to additive constants in the pressures $P_{\rm i}$ and $P_{\rm o}$. This yields Theorem 2.1.

4. Proof of Theorem 2.2

We first prove instability of the trivial solution if $\varrho_{\rm i} > \varrho_{\rm o}$ as claimed in Theorem 2.2. Recall that $\gamma_j := \varrho_j \omega^2/2$.

Theorem 4.1. *If $\varrho_{\rm i} > \varrho_{\rm o}$, then*

$$\partial_t \rho = F(\rho)\,, \quad t > 0\,, \qquad \rho(0) = \rho_0\,,$$

has backward solutions which do exponentially decay to zero. In particular, the trivial solution $\rho = 0$ of this flow is unstable.

Proof. The compact embedding $h^{4+\delta}(\mathbb{S}^1) \hookrightarrow h^{1+\delta}(\mathbb{S}^1)$ and Proposition 3.10 imply that $\partial F(0)$ has compact resolvent. So the spectrum of $\partial F(0)$ consists of eigenvalues only, which, according to Proposition 3.9, are given by $\big\{q_n\,;\, n \in \mathbb{Z} \setminus \{0\}\big\}$ and $q_0 := 0$. Since $\varrho_{\rm i} > \varrho_{\rm o}$, (3.29) implies that $\mu(1) > 0$, hence (3.30) shows that $\operatorname{Re} q_1 > 0$. Clearly,

$$\inf \{\operatorname{Re} q_n\,;\, \operatorname{Re} q_n > 0\} \,>\, 0\,.$$

The assertion now follows from [14, Thm.9.1.3]. □

To prove stability of the trivial solution if $\varrho_{\rm o} > \varrho_{\rm i}$, we need an auxiliary result.

Lemma 4.2. *Let $\mathcal{W}$ be given as in Lemma 3.8. Then*

$$\int_{\mathbb{S}^1} (1+\rho) F(\rho)\, {\rm d}\sigma = 0\,, \quad \rho \in \mathcal{W}\,.$$

Proof. Fix $\rho \in \mathcal{W}$ and set

$$h^{1+\delta}_{0,\rho}(\mathbb{S}^1) := \left\{ f \in h^{1+\delta}(\mathbb{S}^1)\,;\, \int_{\mathbb{S}^1} (1+\rho) f\, {\rm d}\sigma = 0 \right\}\,.$$

We claim that $1 - \mathcal{R}(\rho)$ is an isomorphism on $h^{1+\delta}_{0,\rho}(\mathbb{S}^1)$. To see this, set $T_f := T\big(\rho, S(\rho,f)\big)$ for $f \in h^{1+\delta}(\mathbb{S}^1)$. Then, as in (3.44),

$$\int_{\mathbb{S}^1} (1+\rho) \mathcal{R}(\rho) f\, {\rm d}\sigma = \int_{\mathbb{S}^1} (1+\rho) \mathcal{B}_{\rm o}(\rho) T_f\, {\rm d}\sigma = 0$$

and Lemma 3.8 implies that $1 - \mathcal{R}(\rho)$ is indeed an isomorphism on $h^{1+\delta}_{0,\rho}(\mathbb{S}^1)$. But, as above,

$$\mathcal{B}_{\rm o}(\rho) T\big(\rho, -\mathcal{K}(\rho)\big) \in h^{1+\delta}_{0,\rho}(\mathbb{S}^1)\,.$$

Therefore,

$$F(\rho) = \big(1 - \mathcal{R}(\rho)\big)^{-1} \mathcal{B}_{\rm o}(\rho) T\big(\rho\,,\, -\mathcal{K}(\rho)\big) \in h^{1+\delta}_{0,\rho}(\mathbb{S}^1)\,.$$

□

We conclude the proof of Theorem 2.2 by stating the stability result for which we need to define

$$h^s_0(\mathbb{S}^1) := \big\{ f \in h^s(\mathbb{S}^1)\,;\, \int_{\mathbb{S}^1} f\, {\rm d}\sigma = 0 \big\}\,,$$

for $s > 0$.

Theorem 4.3. *If $\varrho_{\rm o} > \varrho_{\rm i}$, the trivial solution $\rho = 0$ of*

$$\partial_t \rho = F(\rho)\,, \quad t > 0\,,$$

is stable. More precisely, there are numbers $\omega, r, M > 0$ such that for each initial datum $\rho_0 \in h^{1+\delta}_0(\mathbb{S}^1)$ with $\|\rho_0\|_{h^{1+\delta}(\mathbb{S}^1)} \le r$ there is a unique global solution

$$\rho \in C\big(\mathbb{R}^+, h^{4+\delta}_0(\mathbb{S}^1)\big) \cap C^1\big(\mathbb{R}^+, h^{1+\delta}_0(\mathbb{S}^1)\big)$$

with $\rho(0) = \rho_0$ *and*

$$\left\| \rho(t) \left(1 + \frac{\rho(t)}{2}\right) \right\|_{h^{4+\delta}(\mathbb{S}^1)} + \|\dot\rho(t)\,(1+\rho(t))\|_{h^{1+\delta}(\mathbb{S}^1)} \;\le\; M\,e^{-\omega t} \left\| \rho_0 \left(1 + \frac{\rho_0}{2}\right) \right\|_{h^{4+\delta}(\mathbb{S}^1)}, \quad t \ge 0\,.$$

Proof. Letting $\zeta := \rho + \rho^2/2$ for $\rho \in \mathcal{W}$, problem $\partial_t\rho = F(\rho)$, $t > 0$, is equivalent to

$$\partial_t \zeta = G(\zeta)\,, \quad t > 0\,, \tag{4.1}$$

where $G \in C^2\big(Z, h^{1+\delta}(\mathbb{S}^1)\big)$ with $Z := \big\{\zeta = \rho + \rho^2/2\,;\, \rho \in \mathcal{W}\big\}$ is given by

$$G(\zeta) := \sqrt{1+2\zeta}\, F\big(\sqrt{1+2\zeta} - 1\big)\,.$$

Moreover, Lemma 4.2 implies $G \in C^2\big(Z_0, h_0^{1+\delta}(\mathbb{S}^1)\big)$ for $Z_0 := Z \cap h_0^{4+\delta}(\mathbb{S}^1)$. Also,

$$\partial G(0) = \partial F(0) \in \mathcal{L}\big(h_0^{4+\delta}(\mathbb{S}^1), h_0^{1+\delta}(\mathbb{S}^1)\big)\,.$$

Thus $\partial G(0)$ has compact resolvent and its (point) spectrum equals $\big\{q_n\,;\, n \in \mathbb{Z}\setminus\{0\}\big\}$. Now, by (3.29),

$$\operatorname{Re} q_n = \frac{A_n}{A_n^2 + B^2}\,\mu(n)\,, \quad n \in \mathbb{Z}\setminus\{0\}\,,$$

while $\varrho_{\rm o} > \varrho_{\rm i}$ implies that

$$\mu(n) = |n|\big(\sigma - 2(\gamma_{\rm o} - \gamma_{\rm i}) - \sigma n^2\big) \le -2(\gamma_{\rm o} - \gamma_{\rm i}) < 0\,, \quad n \in \mathbb{Z}\setminus\{0\}\,.$$

This, combined with (3.30), shows that the spectrum of $\partial G(0)$ is contained in a half plane $[\operatorname{Re}\lambda \le -\omega]$ for some $\omega > 0$. The assertion now follows from [14, Thm.9.1.2] applied to (4.1). $\square$

Note that our analysis yields an explicit estimate of the exponential decay rate ω in terms of the physical parameters through (3.29).

5. Equilibrium Solutions

To prove Proposition 2.3 let $(\Gamma, P_{\rm i}, P_{\rm o})$ be any equilibrium for (2.6)-(2.9) with $\Gamma \in C^{2+\delta}$ and $P_j \in h^{2+\delta}\big(\Omega^j\big)$ for $j = {\rm i}, {\rm o}$, that is,

$$\Delta P_j = 0 \qquad \text{in } \Omega^j\,, \qquad j = \text{i,o}\,, \tag{5.1}$$

$$P_{\rm i} - P_{\rm o} = \sigma\kappa_\Gamma + (\gamma_{\rm o} - \gamma_{\rm i})|x|^2 \qquad \text{on } \Gamma\,, \tag{5.2}$$

$$\alpha_j \partial_{\nu_\Gamma} P_j + \beta_j \partial_{\tau_\Gamma} P_j = 0 \qquad \text{on } \Gamma\,, \qquad j = \text{i,o}\,, \tag{5.3}$$

$$\alpha_{\rm o}\partial_\nu P_{\rm o} + \beta_{\rm o}\partial_\tau P_{\rm o} = 0 \qquad \text{on } [|x| = R]\,. \tag{5.4}$$

Then

$$\begin{aligned} \Delta P_{\rm i} &= 0 && \text{in } \Omega^{\rm i}\,,\\ \alpha_{\rm i}\partial_{\nu_\Gamma} P_{\rm i} + \beta_{\rm i}\partial_{\tau_\Gamma} P_{\rm i} &= 0 && \text{on } \Gamma\,, \end{aligned}$$

so that Lemma 3.5 implies that $P_{\rm i} \equiv c_{\rm i}$ for some constant $c_{\rm i}$. Moreover, since also

$$\begin{aligned} \Delta P_{\rm o} &= 0 && \text{in } \Omega^{\rm o}\,,\\ \alpha_{\rm o}\partial_{\nu_\Gamma} P_{\rm o} + \beta_{\rm o}\partial_{\tau_\Gamma} P_{\rm o} &= 0 && \text{on } \Gamma\,,\\ \alpha_{\rm o}\partial_\nu P_{\rm o} + \beta_{\rm o}\partial_\tau P_{\rm o} &= 0 && \text{on } R\mathbb{S}^1\,, \end{aligned}$$

a similar Fredholm argument as in Lemma 3.5 shows that $P_{\rm o} \equiv c_{\rm o}$ for some constant $c_{\rm o}$. If $\Gamma = \Gamma_\rho$ for some $\rho \in \mathcal{V}$, then we derive equation (2.15) with $c = c_{\rm i} - c_{\rm o}$ from (5.2) and Lemma 3.3. A bootstrapping argument now shows that ρ is smooth. This proves Proposition 2.3.

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