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Calculus of Variations

Local well-posedness for a quasi-stationary droplet model

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Abstract The moving boundary problem for the contact line evolution of a droplet is studied. Local existence and uniqueness of classical solutions is established.

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1 Introduction

Consider the system

$$\begin{cases} -\Delta u = \lambda & \text{in } \Omega_t,\ \text{for } t > 0, \\ u = 0 & \text{on } \partial\Omega_t,\ \text{for } t > 0, \\ \int_\Omega u\,dx = V_0 > 0 & \text{for } t > 0, \\ V = F(|\nabla u|) & \text{on } \partial\Omega_t,\ \text{for } t > 0, \\ \Omega_t\big|_{t=0} = \Omega_0, \end{cases} \tag{1}$$

which is a widley accepted model [7] describing the quasi-static shape evolution of a liquid droplet of height $u(t,x)$ occupying the region $\Omega_t = [u(t,\cdot) > 0]$ at time $t \geq 0$. The integral condition ensures volume conservation during the evolution and is related to the appearance of the (negative) hydro-static pressure (as a Lagrange multiplier) λ in the first

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equation. The fourth equation relates the contact line speed in normal direction to the steepness $|Du| = -\partial_\nu u$ at contact where $\nu = -\frac{\nabla u}{|\nabla u|}$ is the outward pointing normal to $\Gamma_t = \partial\Omega_t$. While the exact empirical relation F can vary [4,12,13], for the purposes of this paper no qualitative assumptions will be made beyond monotonicity

$$F \in \mathrm{C}^\omega\big(\mathbb{R}, \mathbb{R}\big) \qquad \text{with} \quad F'(r) > 0 \quad \text{for } r > 0\,. \tag{2}$$

The regularity assumption can be relaxed.[1]

While the introduction of the model goes back quite some time [7], not many analytical results can be found in the literature. Numerical experiments on stability were performed in [8] and, subsequently, a numerical scheme based on a boundary integral formulation was proposed by [6]. Later Kim and Glasner [9] proposed a concept of weak generalized solution by means of viscosity solution techniques. Since the problem does not admit a comparison principle, they are forced to consider an approximate problem where λ is kept constant on small time intervals. This way they regain access to a comparison principle and use it to introduce a concept of viscosity solution, before taking the limit and thus effectively removing the approximation step. While their most general result is local in time well-posedness (in the sense of short time existence and uniqueness), they also obtain global existence for solutions satisfying a geometric condition. Later [10] proposed a construction of global in time weak solution via a discrete approach based on barriers and the gradient flow structure of the system. Latter is only given when $F(r) = r^2 - 1$ and thus the results are limited to that case. To the best of our knowledge no result has appeared in the literature concerning the short time existence of classical solutions for (1). This gap is filled by the current paper.

2 Reformulation and main result

A domain fixing transformation will be used to reformulate problem (1) which was first used in [5] and has hence come to be known as the Hanzawa transformation. It is obtained by taking a compact closed C^∞-surface $\Gamma \subset \mathbb{R}^n$ close to $\Gamma_0 = \partial\Omega_0$ and introducing coordinates derived from a foliation of a neighborhood of Γ in $\mathbb{R}^n$. More precisely, let $x \in \Gamma$ and denote by $\nu(x)$ the unit outward normal to Γ at x. Since Γ is smooth and compact, there is a $\Lambda > 0$ such that the surfaces

$$\Gamma_\lambda = \big\{x + \lambda\nu(x)\,\big|\, x \in \Gamma\big\},\ \lambda \in (-\Lambda, \Lambda)$$

are equally smooth, disjoint, and fill a tubular neighborhood Ω_Λ of Γ. It is then possible to parametrize Γ_t (as long as it stays in Ω_Λ) by a function

$$\rho = \rho(t, \cdot) : \Gamma \to \mathbb{R}$$

through

$$\Gamma_\rho = \big\{x + \rho(x)\nu(x)\,\big|\, x \in \Gamma\big\}.$$

As the goal of this paper consists in obtaining classical solutions, it will be convenient to work in so-called little Hölder spaces denoted by $h^\beta(O)$ for an open subset O of $\mathbb{R}^n$ and defined through

$$h^\beta(O) = \overline{r_O\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|_{\mathrm{BUC}^\beta(O)}},$$

[1] An inspection of the proof of the main result shows that existence and uniqueness of classical solutions can be guaranteed for F of class $C^{1+\alpha}$ with $\alpha \in (0, 1)$.

i.e. by the closure of the space of restrictions to O of smooth and rapidly decreasing functions in the norm of $\mathrm{BUC}^{\beta}(O)$. Latter denotes the space of bounded, uniformly Hölder continuous functions[2] on O with norm

$$\|u\|_{\mathrm{BUC}^{\beta}(O)} = \sum_{|\alpha|\leq[\beta]} \|\partial^{\alpha}u\|_{\infty,\bar{O}} + \sum_{|\alpha|=\beta} [\partial^{\alpha}u]_{\beta-[\beta],\bar{O}}$$

where $[\beta]$ is the integer part of β and

$$[\partial^{\alpha}u]_{\beta-[\beta],\bar{O}} := \sup_{x\neq y\in O} \frac{|f(x)-f(y)|}{|x-y|^{\beta-[\beta]}}, \quad u \in \mathrm{C}(O).$$

Given a smooth (compact) manifold M, $h^{\beta}(M)$ is defined (similarly) by localization and using a smooth (finite) atlas of M in the canonical way.

In order to obtain a problem on the fixed domain Ω bounded by Γ, diffeomorphisms from Ω to Ω_t need to be specified. To that end, let

$$\rho \in \mathcal{V} := \mathbb{B}_{\mathrm{C}^1(\Gamma)}(0,a) \cap h^{2+\alpha}(\Gamma), \quad a < \Lambda/4,$$

where $\mathbb{B}_E(x,r)$ stands for the ball with radius r centered at $x \in E$ in the normed space E. Denote by Ω_ρ the domain bounded by Γ_ρ and let $\varphi \in \mathrm{C}^{\infty}\big(\mathbb{R},[0,1]\big)$ be a cutoff function satisfying

$$\varphi(\lambda) = \begin{cases} 1, & |\lambda| < a, \\ 0, & |\lambda| \geq 3a, \end{cases}$$

and, for any $y \in \Omega_\Lambda$, let $\big(X(y),\Lambda(y)\big)$ be the "tubular" coordinates of y, i.e.

$$y = X(y) + \Lambda(y)\nu\big(X(y)\big).$$

Notice that $X \in \mathrm{BUC}^{\infty}(\Omega_\Lambda,\Gamma)$ and $\Lambda \in \mathrm{BUC}^{\infty}\big(\Omega_\Lambda,(-\Lambda,\Lambda)\big)$. Defining

$$\theta_\rho(y) := \begin{cases} X(y) + \big[\Lambda(y) + \varphi\big(\Lambda(y)\big)\rho\big(X(y)\big)\big]\nu\big(X(y)\big), & y \in \Omega_\Lambda, \\ y, & y \notin \Omega_\Lambda, \end{cases} \tag{3}$$

it is easily verified that

$$\theta_\rho \in \mathrm{Diff}^{2+\alpha}(\mathbb{R}^n,\mathbb{R}^n) \cap \mathrm{Diff}^{2+\alpha}(\Omega,\Omega_\rho),$$

and that $\theta_\rho(\Gamma) = \Gamma_\rho$, due to the assumptions on ρ. The original problem can then be rewritten in the new coordinate system. To do so, let θ_ρ^* and θ_*^ρ denote the pull-back and the push-forward maps induced by θ_ρ, respectively, i.e.,

$$\begin{aligned} \theta_\rho^* u &:= u \circ \theta_\rho, \quad u \in \mathrm{BUC}(\Omega_\rho), \\ \theta_*^\rho v &:= v \circ \theta_\rho^{-1}, \quad v \in \mathrm{BUC}(\Omega). \end{aligned}$$

It can be verified (see [5]) that

$$\theta_\rho^* \in \mathrm{Isom}\big(h^{k+\alpha}(\Omega_\rho),h^{k+\alpha}(\Omega)\big) \cap \mathrm{Isom}\big(h^{k+\alpha}(\Gamma_\rho),h^{k+\alpha}(\Gamma)\big)$$

and that $\theta_*^\rho = (\theta_\rho^*)^{-1}$ for any choice of $\rho \in \mathcal{V}$ and $k \in \{0,1,2\}$. The function

$$N_\rho : \Omega_\Lambda \to \mathbb{R}, \quad y \mapsto \Lambda(y) - \rho\big((X(y))\big)$$

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[2] In contrast to $h^{\beta}(O)$ the space $\mathrm{BUC}^{\beta}(O)$ is not separable. This shows particularly that $h^{\beta}(O)$ is a proper subspace of $\mathrm{BUC}^{\beta}(O)$.

leads to the representation $\Gamma_\rho = N_\rho^{-1}(0)$ and thus to the formula

$$\nu_\rho(y) = \frac{\nabla N_\rho(y)}{|\nabla N_\rho(y)|}, \quad y \in N_\rho^{-1}(0),$$

for the unit outward normal to Γ_ρ at the point y. The normal velocity V of the front $\Gamma_{\rho(t,\cdot)}$ is given by

$$V(y,t) = -\frac{\partial_t N_\rho(t,y)}{|\nabla N_\rho(y)|} = \frac{\partial_t \rho\big(X(y),t\big)}{|\nabla N_\rho(y)|}, \quad y \in \Gamma_\rho,\ t \in [0,T],$$

for fixed

$$\rho \in \mathrm{C}\big([0,T],\mathcal{V}\big) \cap \mathrm{C}^1\big([0,t],h^{1+\alpha}(\Gamma)\big).$$

Definition 2.1 (i) A family of domains $\{\Omega_t \mid t \in [0,T]\}$ with $h^{2+\alpha}$-boundary Γ_t together with a family of functions

$$u(t,\cdot) \in h^{2+\alpha}(\Omega_t), \quad t \in [0,T],$$

is called a classical $h^{2+\alpha}$-solution of (1) if all equations are satisfied pointwise.

(ii) A pair (v,ρ) such that

$$v \in \mathrm{C}\big([0,t],h^{2+\alpha}(\Omega)\big) \cap \mathrm{C}^1\big([0,T],h^{\alpha}(\Omega)\big) \text{ and}$$
$$\rho \in \mathrm{C}\big([0,T],\mathcal{V}\big) \cap \mathrm{C}^1\big([0,T],h^{1+\alpha}(\Gamma)\big),$$

is called classical $h^{2+\alpha}$-solution of

$$\begin{cases} -\theta_\rho^* \Delta \theta_*^\rho v = \lambda & \text{in } (0,T)\times\Omega, \\ v = 0 & \text{on } (0,T)\times\partial\Omega, \\ \int_\Omega v(y)\big|D\theta_\rho(y)\big|\,dy = V_0, & \\ \partial_t \rho = \theta_\rho^* |\nabla N_\rho| F\Big[-\theta_\rho^* \frac{\nabla\theta_*^\rho v\cdot\nabla N_\rho}{|\nabla N_\rho|}\Big] & \text{on } (0,T)\times\Gamma, \\ \rho(0,\cdot) = \rho_0 & \text{on } \Gamma, \end{cases} \tag{4}$$

if it satisfies the system in the pointwise sense.

Proposition 2.2 *Problems* (1) *and* (4) *are equivalent in that any pair* (v,p) *with*

$$v \in \mathrm{C}\big([0,t],h^{2+\alpha}(\Omega)\big) \cap \mathrm{C}^1\big([0,T],h^{\alpha}(\Omega)\big),$$
$$p \in \mathrm{C}\big([0,T],\mathcal{V}\big) \cap \mathrm{C}^1\big([0,T],h^{1+\alpha}(\Gamma)\big)$$

is a classical $h^{2+\alpha}$*-solution of* (4) *if and only if the domains* $\Omega_{\rho(t,\cdot)}$ *bounded by* $\Gamma_{\rho(t,\cdot)}$ *and the functions* $u := \theta_*^\rho v \in h^{2+\alpha}\big(\Omega_{\rho(t,\cdot)}\big)$ *yield a solution of* (1).

Proof Observe that $\partial_{\nu_\rho} u < 0$ for any smooth solution of (1) by the strong maximum principle and that

$$-|Du| = \nabla u \cdot \left(-\frac{\nabla u}{|\nabla u|}\right) = \partial_{\nu_\rho} u,$$

since Γ_t is the zero level set of the nonnegative function u. It follows that

$$\frac{\theta_*^\rho \partial_t \rho}{|\nabla N_\rho|} = F\big(|Du|\big) = F\big(-\partial_{\nu_\rho} u\big) = F\left(-\theta_\rho^* \frac{\nabla\theta_*^\rho v \cdot \nabla N_\rho}{|\nabla N_\rho|}\right),$$

if $v = \theta_\rho^* u$. This can also be rewritten as

$$\partial_t \rho = \theta_\rho^* |\nabla N_\rho| F \left(-\theta_\rho^* \frac{\nabla \theta_*^\rho v \cdot \nabla N_\rho}{|\nabla N_\rho|} \right).$$

The rest follows from the regularity properties of the diffeomorphism θ_ρ. □

In the next step system (4) is reduced to a single evolution equation by solving the time dependent elliptic boundary value problem

$$\begin{cases} -\mathbb{A}(\rho)v = \lambda & \text{in } (0,T)\times\Omega, \\ v = 0 & \text{on } (0,T)\times\Gamma, \\ I(\rho)v = V_0 & \text{on } [0,T], \end{cases} \tag{5}$$

obtained by fixing $\rho \in \mathrm{C}\big([0,T],\mathcal{V}\big) \cap \mathrm{C}^1\big([0,T], h^{1+\alpha}(\Gamma)\big)$. The short-hand

$$\mathbb{A}(\rho)v := -\theta_\rho^* \Delta \theta_*^\rho v,$$

$$I(\rho)v := \int_\Omega v(y)|D\theta_\rho(y)|\, dy,$$

are used to simplify the notation. The operator given by

$$\mathbb{B}(\rho)v = -\gamma_\Gamma \theta_\rho^* \big(\nabla \theta_*^\rho v \big| \nabla N_\rho\big), \tag{6}$$

where γ_Γ is the restriction operator to the boundary, will also be needed later on.

Lemma 2.3 *The above operators all depend analytically on ρ, i.e.*

$$(\mathbb{A},\mathbb{B}) \in \mathrm{C}^\omega\Big(\mathcal{V}, \mathcal{L}\big(h^{2+\alpha}(\Omega), h^\alpha(\Omega)\times h^{1+\alpha}(\Gamma)\big)\Big), \tag{7}$$

$$I \in \mathrm{C}^\omega(\mathcal{V},\mathbb{R}), \tag{8}$$

$$[\rho \mapsto |\nabla N_\rho|] \in \mathrm{C}^\omega\big(\mathcal{V}, h^{1+\alpha}(\Gamma)\big). \tag{9}$$

Proof Let dx^2 denote the Euclidean metric on $\mathbb{R}^n$ and $\theta_\rho^* dx^2$ the pull-back metric on $\overline{\Omega}$, i.e.,

$$\theta_\rho^* dx^2\big|_x(\xi,\eta) := dx^2\big|_{\theta_\rho(x)}\big(d_x\theta_\rho(\xi)\big|d_x\theta_\rho(\eta)\big), \quad x \in \overline{\Omega},\ \xi,\eta \in \mathrm{T}_x(\overline{\Omega}).$$

It follows that $\mathbb{A}(\rho)$ and $\mathbb{B}(\rho)$ are the Laplace–Beltrami operator and the unit outward normal derivative of $(\Omega, \theta_\rho^* dx^2)$, respectively. The metric depends analytically on ρ as do $|D\theta_\rho|$ and $|\nabla N_\rho|$. All maps are therefore analytic, since θ_ρ depends analytically (algebraically, in fact) on ρ. □

Lemma 2.4 *The boundary value problem*

$$\begin{cases} -\mathbb{A}(\rho)v = \lambda f & \textit{in } (0,T)\times\Omega, \\ v = 0 & \textit{on } (0,T)\times\Gamma, \\ I(\rho)v = V_0 & \textit{on } [0,T], \end{cases} \tag{10}$$

can be solved uniquely for any given $f \in h^\alpha(\Omega)$ and $V_0 \in \mathbb{R}$. The solution is given by

$$v = \frac{V_0}{I(\rho)\tilde{\mathbb{S}}(\rho)f} \cdot \tilde{\mathbb{S}}(\rho)f, \qquad \lambda = \frac{V_0}{I(\rho)\tilde{\mathbb{S}}(\rho)f},$$

where $\tilde{\mathbb{S}}(\rho)f$ is the unique solution of

$$\begin{cases} -\mathbb{A}(\rho)v = f & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma. \end{cases}$$

It holds that

$$\tilde{\mathbb{S}}(\rho) \in \mathrm{C}^\omega\left(\mathcal{V}, \mathcal{L}(h^\alpha(\Omega), h^{2+\alpha}(\Omega))\right) \tag{11}$$

and there is a $C = C(\rho)$ such that

$$\|\tilde{\mathbb{S}}(\rho)f\|_{2+\alpha,\Omega} \le C\|f\|_{\alpha,\Omega}$$

Proof The results follow from classical regularity results for elliptic boundary value problems and the fact that the additional degree of freedom introduced by λ is compensated by the integral condition. □

It is convenient to introduce some further notation:

$$\mathbb{S}(\rho) := \lambda(\rho) \cdot \tilde{\mathbb{S}}(\rho), \qquad \rho \in \mathcal{V}.$$

Remark 2.5 The strong maximum principle implies that, given $f \ge 0$ and $V_0 > 0$, we have

$$\partial_{\nu_\rho}\left(\theta_*^\rho\, \mathbb{S}(\rho)f\right) < 0 \quad \text{on} \quad \Gamma_\rho$$

since the solution is positive in Ω_ρ.

Problem (4) therefore reduces to

$$\begin{cases} \partial_t \rho = \theta_\rho^* |\nabla N_\rho| \cdot F\left(-\theta_\rho^* \frac{1}{|\nabla N_\rho|}\mathbb{B}(\rho)\mathbb{S}(\rho)\mathbf{1}\right) & \text{on } [0,T] \times \Gamma, \\ \rho(0,\cdot) = \rho_0 \in \mathcal{V}, \end{cases} \tag{12}$$

where $\mathbf{1}(x) \equiv 1,\ x \in \Omega$. To shorten the notation, let

$$\Phi(\rho) := -\theta_\rho^* |\nabla N_\rho| \cdot F\left(-\theta_\rho^* \frac{1}{|\nabla N_\rho|}\mathbb{B}(\rho)\mathbb{S}(\rho)\mathbf{1}\right), \quad \rho \in \mathcal{V}. \tag{13}$$

It can be seen from Lemma 2.3, (2), and (11) that

$$\Phi \in \mathrm{C}^\omega\left(\mathcal{V}, h^{1+\alpha}(\Gamma)\right). \tag{14}$$

Definition 2.6 A function $\rho \in \mathrm{C}\big([0,T],\mathcal{V}\big) \cap \mathrm{C}^1\big([0,T], h^{1+\alpha}(\Gamma)\big)$ is called a classical Hölder solution of (12) it it satisfies the equation pointwise on $[0,T]$.

Lemma 2.7 *The function ρ is a classical Hölder solution of* (12) *if and only if* $\big(\mathbb{S}(\rho)\mathbf{1}, \rho\big)$ *is a classical Hölder solution of* (4).

Evolution equation (12) will be analyzed by means of linearization and maximal regularity in the sense of Da Prato and Grisvard, cf. [11]. For this, a good understanding of the Fréchet derivative of Φ is required. This analysis will be carried out in the next section. In fact, Theorem 4.1 below and the characterization of small Hölder spaces as continuous interploation spaces allows us to apply Théorème 4.1 in [11], cf. also Theorem 2.7 and Theorem 2.14 in [2]. As a consequence we find the following result.

Theorem 2.8 *Given $\rho_0 \in \mathcal{V}$, there exists $T > 0$ and a unique solution*

$$\rho \in \mathrm{C}\big([0,T],\mathcal{V}\big) \cap \mathrm{C}^1\big([0,T], h^{1+\alpha}(\Gamma)\big)$$

of (12) *with $\rho(0,\cdot) = \rho_0$. This implies the existence of a unique classical Hölder solution of* (4) *and* (1) *on* $[0,T]$*, as well.*

3 Linearization

In this section we compute the linearization of the operator Φ from 14. Its structure will be analyzed in the next section. Since we already verified that Φ is smooth, will use Gâteaux derivatives to find the Fréchet derivative $\partial\Phi$. Let $\rho \in \mathcal{V}$ be given. In order to compute the linearization of (13), the following terms need to be considered

$$\theta_\rho^*|\nabla N_\rho|,\ \mathbb{B}(\rho),\ \tilde{\mathbb{S}}(\rho),\ I(\rho).$$

Starting from the first term notice that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\theta^*_{\rho+\varepsilon h}|\nabla N_{\rho+\varepsilon h}| = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\theta^*_{\rho+\varepsilon h}|\nabla N_\rho| + \theta^*_\rho\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}|\nabla N_{\rho+\varepsilon h}| =: \mathrm{I}+\mathrm{II}.$$

Recall that

$$N_\rho(x) = d(x,\Gamma) - \rho\big(\mathrm{pr}_\Gamma(x)\big) \equiv \Lambda(x) - \rho\circ X(x),$$

so that

$$\mathrm{II} = -\theta^*_\rho\frac{\nabla N_\rho}{|\nabla N_\rho|}\cdot\nabla(h\circ X) = -\big(\theta^*_\rho\partial_{\nu_\rho}X\big)d_{X(\cdot)}h$$

where $d_y f$ denotes the tangential of f at $y\in\Gamma$. As an operator acting on h this amounts to a (tangential) differential operator (on Γ). As for I, we recall the definition (3) of θ_ρ together with the fact that $\theta_\rho(\Gamma)=\Gamma_\rho$. Let $u\in h^{2+\alpha}(\Omega_\rho)$ and consider

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\theta^*_{\rho+\varepsilon h}u = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big[u(\theta_{\rho+\varepsilon h}) - u(\theta_\rho)\big].$$

Notice that

$$\theta_{\rho+\varepsilon h} - \theta_\rho = \varepsilon h\big(X(y)\big)\varphi\big(\Lambda(y)\big)\nu\big(X(y)\big) = \varepsilon h\big(X(y)\big)\nu\big(X(y)\big),$$

in a neighborhood of Γ_ρ. It follows that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\theta^*_{\rho+\varepsilon h}u = \big(\theta^*_\rho\partial_\nu u\big)h\circ X. \tag{15}$$

Then

$$\mathrm{I} = \big(\theta^*_\rho\partial_\nu|\nabla N_\rho|\big)h\circ X,$$

which requires the computation of $\partial_\nu|\nabla N_\rho|$. Notice first that

$$\nabla N_\rho(x) = -\sum_{j=1}^{n-1}\partial_{\tau_j}\rho\big(\mathrm{pr}_\Gamma(x)\big)\tau_j + 1\,\nu\big(\mathrm{pr}_\Gamma(x)\big),$$

if the gradient is computed in a local orthogonal basis $\tau_1,\dots,\tau_{n-1}$ for the tangent space to Γ at $\mathrm{pr}_\Gamma(x)$ completed to a basis of $\mathbb{R}^n$ by means of the normal ν to Γ at the same point. It is apparent that ∇N_ρ does not vary in the ν-direction as the only non-trivial dependence is in the tangential directions. Therefore one has that $\partial_\nu|\nabla N_\rho|=0$. Summarizing, we get the following relation:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\theta^*_{\rho+\varepsilon h}|\nabla N_{\rho+\varepsilon h}| = -\big(\theta^*_\rho\partial_{\nu_\rho}X\big)d_{X(\cdot)}h,\qquad h\in h^{2+\alpha}(\Gamma). \tag{16}$$

Next recall that

$$\mathbb{S}(\rho)\mathbf{1} = \lambda(\rho)\tilde{\mathbb{S}}(\rho)\mathbf{1} \quad \text{for} \quad \lambda(\rho) := \frac{V_0}{I(\rho)\tilde{\mathbb{S}}(\rho)\mathbf{1}},$$

so that

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathbb{B}(\rho+\varepsilon h)\mathbb{S}(\rho+\varepsilon h)\mathbf{1} \\
&= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathbb{B}(\rho+\varepsilon h)\mathbb{S}(\rho)\mathbf{1} + \mathbb{B}(\rho)\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\lambda(\rho+\varepsilon h)\tilde{\mathbb{S}}(\rho)\mathbf{1} + \mathbb{B}(\rho)\lambda(\rho)\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde{\mathbb{S}}(\rho+\varepsilon h)\mathbf{1} \\
&=: \text{III+IV+V}.
\end{aligned}$$

As for III for now just introduce the following notation which will be useful later

$$\mathbb{M}(\rho) := \partial\mathbb{B}(\rho)\big[\,\cdot\,, \mathbb{S}(\rho)\mathbf{1}\big] \in \mathcal{L}\big(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)\big)$$

where it is thought of $\mathbb{M}(\rho)$ as defining a linear operator acting on h so that the interpretation of the inclusion ought to be clear. As for V let

$$G\big(\rho, \tilde{\mathbb{S}}(\rho)\mathbf{1}\big) := \big(\mathbb{A}(\rho)\tilde{\mathbb{S}}(\rho)\mathbf{1}, \gamma_\Gamma\tilde{\mathbb{S}}(\rho)\mathbf{1}\big),$$

and notice that $G\big(\rho, \tilde{\mathbb{S}}(\rho)\mathbf{1}\big) = (\mathbf{1}, 0)$. Thus we get

$$0 \equiv \partial_1 G\big(\rho, \tilde{\mathbb{S}}(\rho)\mathbf{1}\big)h + \partial_2 G\big(\rho, \tilde{\mathbb{S}}(\rho)\mathbf{1}\big)\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde{\mathbb{S}}(\rho+\varepsilon h)\mathbf{1}.$$

Latter implies that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde{\mathbb{S}}(\rho+\varepsilon h)\mathbf{1} = -\tilde{\mathbb{S}}(\rho)\partial\mathbb{A}(\rho)\big[h, \tilde{\mathbb{S}}(\rho)\mathbf{1}\big],$$

where $\partial\mathbb{A}(\rho)[h, v] := \frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathbb{A}(\rho+\varepsilon h)v$. Again it is useful to introduce the notation

$$\mathbb{K}(\rho) := \partial\mathbb{A}(\rho)[\,\cdot\,, \mathbb{S}(\rho)\mathbf{1}] \in \mathcal{L}\big(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)\big).$$

The only remaining term is

$$\text{IV} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\lambda(\rho+\varepsilon h)\mathbb{B}(\rho)\tilde{\mathbb{S}}(\rho)\mathbf{1}$$

since λ is real-valued. Recalling the definition of λ, it follows that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\lambda(\rho+\varepsilon h) = \frac{V_0}{\big(I(\rho)\tilde{\mathbb{S}}(\rho)\mathbf{1}\big)^2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_\Omega \tilde{\mathbb{S}}(\rho+\varepsilon h)\mathbf{1}\cdot|D\theta_{\rho+\varepsilon h}|\,dx.$$

Clearly

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_\Omega \tilde{\mathbb{S}}(\rho+\varepsilon h)\mathbf{1}\cdot|D\theta_{\rho+\varepsilon h}|\,dx \\
&= \int_\Omega \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\tilde{\mathbb{S}}(\rho+\varepsilon h)\mathbf{1}\cdot|D\theta_\rho|\,dx + \int_\Omega \tilde{\mathbb{S}}(\rho)\mathbf{1}\cdot\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}|D\theta_{\rho+\varepsilon h}|\,dx \\
&= -\int_\Omega \tilde{\mathbb{S}}(\rho)\mathbb{K}(\rho)h\cdot|D\theta_\rho|\,dx + \int_\Omega \tilde{\mathbb{S}}(\rho)\mathbf{1}\cdot\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}|D\theta_{\rho+\varepsilon h}|\,dx.
\end{aligned}$$

The first term after the last equality sign amounts to a (nonlocal) rank 1 (and thus compact) operator in $\mathcal{L}\big(h^{2+\alpha}(\Gamma),\mathbb{R}\big)$ acting on h. As for the second term, it is easily checked with the help of Liouville's theorem that

$$\begin{aligned}&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}|D\theta_{\rho+\varepsilon h}|\\&=|D\theta_\rho|\,\mathrm{trace}\big(D\theta_\rho^{-1}\big)\Big[\nabla(h\circ X)\otimes D(\varphi\circ\Lambda\,\nu\circ X)+(h\circ X)D(\varphi\circ\Lambda\,\nu\circ X)\Big],\end{aligned}$$

and it therefore acts as a (nonlocal) rank 1 operator in $\mathcal{L}\big(h^{1+\alpha}(\Gamma),\mathbb{R}\big)$. Letting

$$L(\rho)h:=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\lambda(\rho+\varepsilon h),\quad h\in h^{2+\alpha}(\Gamma),$$

the above computations can be summarized by

Lemma 3.1 *Given $\rho\in\mathcal{V}$, the operator $L(\rho)$ belongs to $\mathcal{L}\big(h^{2+\alpha}(\Gamma),\mathbb{R}\big)$ and, therefore, $\mathbb{L}(\rho)$ defined by*

$$\mathbb{L}(\rho)h=\big(L(\rho)h\big)\cdot\mathbb{B}(\rho)\tilde{\mathbb{S}}(\rho)\mathbf{1}$$

is of rank 1.

To further unravel the structure of the linearization $\partial\Phi(\rho)$, it is useful to introduce the following nonlinear composition operators:

$$P(\rho):=\theta_\rho^*|\nabla N_\rho|,\qquad Q(\rho):=\mathbb{B}(\rho)\mathbb{S}(\rho)\mathbf{1},\quad \rho\in\mathcal{V}.$$

The next result will be needed later on.

Lemma 3.2 *$P,\ Q\in C^\omega\left(\mathcal{V},h^{1+\alpha}(\Gamma)\right)$ and, given $\rho\in\mathcal{V}$, the functions $P(\rho)$ and $-Q(\rho)$ are positive on Γ.*

Proof The first assertion follows from Lemma 2.3 and (11). Clearly $P(\rho):=\theta_\rho^*|\nabla N_\rho|$ is positive on Γ and the relation

$$-\mathbb{B}(\rho)\mathbb{S}(\rho)\mathbf{1}=-P(\rho)\cdot\theta_\rho^*\partial_{\nu_\rho}\left(\theta_*^\rho\,\mathbb{S}(\rho)\mathbf{1}\right)$$

implies the positivity of $-Q(\rho)$ on Γ in view of Remark 2.5. □

Based on Lemma 3.2 we introduce further notation.

$$F_0(\rho):=F(-Q(\rho)/P(\rho)),\qquad F_1(\rho):=F'(-Q(\rho)/P(\rho)),\quad \rho\in\mathcal{V}.$$

Lemma 3.2 and assumption (2) imply that

$$F_0,\ F_1\in C^\omega\left(\mathcal{V},h^{1+\alpha}(\Gamma)\right)$$

and that

$$F_1(\rho)>0\quad\text{on}\quad\Gamma\quad\text{for any }\rho\in\mathcal{V}. \tag{17}$$

Finally, let

$$\mathbb{D}_1(\rho):=\partial P(\rho),\qquad \mathbb{D}_{-1}(\rho):=\partial\big(1/P(\rho)\big),$$

where ∂ again indicates the Fréchet derivative. Using this notation, we have

$$\begin{aligned}\partial\Phi(\rho)=F_1(\rho)\cdot\lambda(\rho)\cdot\mathbb{B}(\rho)\circ\tilde{\mathbb{S}}(\rho)\circ\mathbb{K}(\rho)+F_1(\rho)\cdot\mathbb{M}(\rho)-F_0(\rho)\cdot\mathbb{D}_1(\rho)\\+P(\rho)\cdot F_1(\rho)\cdot Q(\rho)\cdot\mathbb{D}_{-1}(\rho)+F_1(\rho)\cdot\mathbb{L}(\rho).\end{aligned} \tag{18}$$

Lemma 3.3 *Given $\rho \in \mathcal{V}$, each of the operators $\mathbb{D}_1(\rho)$, $\mathbb{D}_{-1}(\rho)$, and $\mathbb{M}(\rho)$ is a first order differential operator on Γ with coefficients depending on ρ, which are of class $h^{1+\alpha}(\Gamma)$. Moreover,*

$$\mathbb{D}_1,\ \mathbb{D}_{-1}, \mathbb{M} \in C^\omega\left(\mathcal{V}, \mathcal{L}(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)\right).$$

Proof The assertions on $\mathbb{D}_1(\rho)$ follow from Lemma 3.2 and (16); those on $\mathbb{D}_{-1}(\rho)$ can be derived similarly. Invoking Lemma 2.3, (6), (15) the assertions concerning $\mathbb{M}$ follow easily. □

4 Localization

The focus of this section is to outline the structure of $\partial\Phi(\rho)$. Consider a point $x \in \Gamma$. By introducing local coordinates in tangential and normal direction to Γ in a tubular neighborhood $O_\Lambda \subset \Omega_\Lambda$ of x, it can be assumed without loss of generality that

$$\Gamma = \mathbb{R}^{n-1} \times \{0\},\ \Omega = \mathbb{H}^n_- = [x_n < 0],$$

and that

$$\Omega_\rho = [x_n < \rho(x')],\ \Gamma_\rho = [x_n = \rho(x')],\ x = (x', x_n).$$

Then the map θ_ρ is simply given by

$$\theta_\rho : \Omega \to \Omega_\rho,\ x = (x', x_n) \mapsto \big(x', x_n + \rho(x')\big) = y = (y', y_n).$$

For $v : \Omega \to \mathbb{R}$ it is easily seen that

$$\mathbb{A}(\rho)v = \theta^*_\rho \Delta_y \theta^\rho_* v = \sum_{j=1}^n \alpha^\rho_j(x')\partial^2_{x_j} v - 2\nabla\rho \cdot \nabla_{x'}\partial_n v - \Delta\rho(x')\partial_{x_n} v,$$

for $\alpha^\rho_j \equiv 1$ if $j = 1, \dots, n-1$ and $\alpha^\rho_n = 1 + |\nabla\rho|^2$. While the structure and properties of the differentiations involved can be more readily read off the above representation of $\mathbb{A}(\rho)$, a computation performed in [5] shows that the corresponding expression in the general coordinates considered for the Hanzawa transformation would read

$$\mathbb{A}(\rho)v = -\sum_{j,k=1}^n a_{jk}(\rho, \nabla\rho)\partial_{x_j x_k} v + \sum_{j=1}^n a_j(\rho, \nabla\rho)\partial_{x_j} v + (W\rho)\,\partial_{x_n} v,$$

where the last direction points along $\nu_\Gamma(x)$ and $a_{jk}, a_j \in C^\infty$. The above notation for the coefficients $a_{jk}(\rho, \nabla\rho)$ has to be understood in the sense of substitution operators. This means that there are smooth functions $\tilde{a}_{jk}$ in $n+1$ variables such that $a_{jk}(\rho, \nabla\rho)(x', x_n) = \tilde{a}_{jk}(\rho(x'), \nabla\rho(x'))$, where (x', x_n) indicates local coordinates . Finally, W is a second order uniformly elliptic differential operator of the tangential variables. For later pruposes we denote a representation of W in local coordinates by

$$Wh = \sum_{j,k=1}^{n-1} w^{jk}\partial_j\partial_k, \quad h \in h^{2+\alpha}(\Gamma). \tag{19}$$

It may be worth mentioning, that the coefficients w^{jk} depend only on the geometry of Γ, but are otherwise independent of ρ. The precise calculation has been carried out in [5] and we refer the interested reader to that paper. Similarly it can be checked that

$$\mathbb{B}(\rho) = \sum_{j=1}^{n} b_j(\rho, \nabla\rho)\partial_{x_j}$$

in local coordinates.

Given two Banach spaces E_1 and E_0 such that E_1 is dense and continuously injected in E_0, let $\mathcal{H}(E_1, E_0)$ denote the space of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as an unbounded operator on E_0, generates a strongly continuous analytic semigroup on E_0. It follows from well-known perturbation results for analytic semigroups that $\mathcal{H}(E_1, E_0)$ is an open subset of $\mathcal{L}(E_1, E_0)$ (see for instance [1, Thm.1.3.1]). On $\mathcal{H}(E_1, E_0)$ we always use the relative topology induced by $\mathcal{L}(E_1, E_0)$.

Given any $\rho \in \mathcal{V}$, it follows from Theorem 5.2 in [5] that the operator

$$\mathbb{G}(\rho) := \mathbb{B}(\rho) \circ \tilde{\mathbb{S}}(\rho) \circ \mathbb{K}(\rho) + \mathbb{M}(\rho)$$

belongs to $\mathcal{H}(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma))$. The method of proof is to associate to $\mathbb{G}(\rho)$ a Fourier multiplication operator $\mathcal{G}$ acting on $h^{1+\alpha}(\Gamma)$, to prove that $\mathcal{G} \in \mathcal{H}(h^{2+\alpha}(\mathbb{R}^{n-1}), h^{1+\alpha}(\mathbb{R}^{n-1}))$ and to use sharp perturbation results for the class $\mathcal{H}(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma))$ based on estimates of the form

$$\|\mu_l^*(\psi_l \mathbb{G}(\rho)h) - \mathcal{G}\mu_l^*(\psi_l h)\|_{h^{1+\alpha}} \le \varepsilon \|\mu_l^*(\psi_l h)\|_{h^{2+\alpha}} + C\|h\|_{h^{2+\beta}}, \quad h \in h^{1+\alpha}(\Gamma). \tag{20}$$

Here $\beta \in (0, \alpha)$ is fixed, $\varepsilon > 0$ is arbitrarily small, and $C > 0$. To explain further the notation used in (20), we start with the Fourier multiplication operator $\mathcal{G}$. The construction of $\mathcal{G}$ is based on the choice of a suitable partition of unity $\{(U_l, \psi_l)\,;\ 1 \le l \le m\}$ of a tubular neighbourhood of Γ and by freezing the coefficients of $(\mathbb{A}(\rho), \mathbb{B}(\rho))$ at the point $0 \in \mathbb{R}^{n-1}$. To be more precise, let $s_l \in C^\infty\left((-\delta, \delta)^{n-1}, U_l\right)$ with $\delta > 0$ be a parametrization of $U_l \cap \Gamma$ and set

$$\mu_l : (-\delta, \delta)^{n-1} \times [0, \delta) \to U_l, \qquad (\omega, r) \mapsto s_l(\omega) - r\nu_\Gamma(s_l(\omega)).$$

We denote the pull-back operator corresponding to the parametrization μ_l by μ_l^*, i.e. $\mu_l^* h := h \circ \mu_l$. In order to unburden the notation we drop the index $l \in \{1, \ldots, m\}$ in the following and set $\hat{\rho} := \rho \circ \mu$, as well as

$$a_{jk}^0 := a_{jk}(\hat{\rho}, \nabla\hat{\rho})(0), \quad b_j^0 := b_j(\hat{\rho}, \nabla\hat{\rho})(0), \quad 1 \le j, k \le n. \tag{21}$$

The symbol of the above mentioned Fourier operator $\mathcal{G}$ splits naturally in two parts $g_1(\xi)$ and $g_2(\xi)$, stemming form the operators $\mathbb{B}(\rho)\tilde{\mathbb{S}}(\rho)\mathbb{K}(\rho)$ and $\mathbb{M}(\rho)$, respectively. To give the precise expressions for $g_1(\xi)$ and $g_2(\xi)$, we need some notation. Let

$$\mathbf{a} := \left(a_{1n}^0, \ldots, a_{(n-1)n}^0\right) \quad \text{and} \quad a_0 := -\sum_{j,k=1}^{n-1} a_{jk}^0 \xi^j \xi^k, \quad \xi \in \mathbb{R}^{n-1}$$

and set

$$\gamma(\xi) := \frac{i(\mathbf{a}|\xi)}{a_{nn}^0} + \frac{1}{a_{nn}^0}\sqrt{a_{nn}^0\left[1 + a_0(\xi)\right] - (\mathbf{a}|\xi)^2}.$$

Note that for $\xi \in \mathbb{R}^{n-1}$ fixed, $\gamma(\xi)$ is the unique root with positive real part of the quadratic polynomial

$$q_\xi(z) := 1 + a_0(\xi) + 2i(\mathbf{a}|\xi)z - a_{nn}^0 z^2.$$

in the variable $z \in \mathbb{C}$. It should be remarked that q_ξ is the characteritic polynomial of the second order ordinary differential equation for $\hat{u}(\xi, \cdot) = \mathcal{F}u(\xi, \cdot)$, the partial Fourier transform[3] of the solution u of

$$u + \sum_{j,k=1}^{n} a^0_{jk}\partial_j\partial_k u = 0 \quad \text{on} \quad \mathbb{R}^{n-1} \times (0, \infty).$$

Finally, we need

$$b^0_j := b_j(\rho, \nabla\rho)(0), \quad \mathbf{b} := \big(b^0_1, \dots, b^0_{n-1}\big),$$

for $1 \le j \le n$, (21), and

$$w_0^{jk} := w^{jk}(0), \quad 1 \le j,\, k \le n-1,$$

cf. (19). After these preparations, let

$$g_1(\xi) := b^0_n \cdot \gamma(\xi) \cdot \frac{1 + \sum_{j,k=1}^{n-1} w_0^{jk}\xi_j\xi_k}{1 + a_0(\xi)}, \quad \xi \in \mathbb{R}^{n-1}.$$

Then g_1 is the symbol of the Fourier multiplication operator associated to $\mathbb{B}(\rho)\tilde{\mathbb{S}}(\rho)\mathbb{K}(\rho)$. To give the symbol g_2 of the second addend, notice that there are smooth functions

$$m_j \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^{n-1}), \quad j \in \{1, \dots, n-1\}$$

such that

$$\mathbb{M}(\rho)h = \sum_{j=1}^{n-1} m_j(\rho, \nabla\rho)\partial_j h, \quad h \in h^{2+\alpha}(\Gamma). \tag{22}$$

Letting now $m^0_j := m_j(\rho, \nabla\rho)(0)$ and

$$g_2(\xi) := i\sum_{j=1}^{n-1} m^0_j\xi_j, \qquad \xi \in \mathbb{R}^{n-1},$$

we have

$$\mathcal{G} = \mathcal{F}^{-1}(g_1 + g_2)\mathcal{F}.$$

It has been shown in Theorem 4.2 in [5] that there exists $C > 0$ such that

$$g_1(\xi) \ge C \cdot b^0_n \cdot \sqrt{1 + |\xi|^2} \quad \text{for all } \xi \in \mathbb{R}^{n-1}. \tag{23}$$

This estimate is the crucial ingredient needed to apply the Mikhlin-Hörmander multiplier theorem and establish that

$$\mathcal{G} \in \mathcal{H}\big(h^{2+\alpha}(\mathbb{R}^{n-1}), h^{1+\alpha}(\mathbb{R}^{n-1})\big),$$

cf. the proof of Theorem 4.2 in [5]. It is worth mentioning that no assumption on g_2 is needed in this procedure, since g_2 is purely imaginary.

Theorem 4.1 $\Phi \in \mathrm{C}^\omega\Big(\mathcal{V}, \mathcal{H}\big(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)\big)\Big)$.

[3] Throughout this paper $\mathcal{F}$ stands for the Fourier transform on $\mathbb{R}^{n-1}$.

Proof (i) In view of (14) it suffices to show that

$$\partial\Phi(\rho) \in \mathcal{H}\big(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)\big)$$

for given $\rho \in \mathcal{V}$.

(ii) In order to investigate the first term on the right-hand side of (18), let

$$F_1^0 := F_1(\hat{\rho})(0), \quad \lambda_0 := \lambda(\hat{\rho}),$$

and notice that $F_1^0 > 0$ and $\lambda_0 > 0$ by (17) and the assumption $V_0 > 0$, respectively. Hence, letting again $g_1(\xi)$ denote the symbol of the Fourier multiplication operator induced by

$$\mathbb{G}_F(\rho) := F_1(\rho) \cdot \lambda(\rho) \cdot \mathbb{B}(\rho) \circ \tilde{\mathbb{S}}(\rho) \circ \mathbb{K}(\rho),$$

we get from (23) the estimate

$$g_1(\xi) \geq C \cdot F_1^0 \cdot \lambda_0 \cdot b_n^0 \cdot \sqrt{1+|\xi|^2} \quad \text{for all} \quad \xi \in \mathbb{R}^{n-1},$$

from which it can be concluded that $\mathcal{F}^{-1} g_1 \mathcal{F} \in \mathcal{H}(h^{2+\alpha}(\mathbb{R}^{n-1}), h^{1+\alpha}(\mathbb{R}^{n-1}))$, cf. the proof of [5, Thm4.2]. Pertubation arguments, based on (20) then imply that

$$\mathbb{G}_F(\rho) \in \mathcal{H}(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)),$$

cf. Lemma 5.1 and the proof of Theorem 5.2 in [5]

(iii) It follows from Lemma 3.3 that each of the operators

$$F_1(\rho) \cdot \mathbb{M}(\rho), \ -F_0(\rho) \cdot \mathbb{D}_1(\rho), \ P(\rho) \cdot F_1(\rho) \cdot Q(\rho) \cdot \mathbb{D}_{-1}(\rho)$$

is a first order differential operator acting on $h^{2+\alpha}(\Gamma)$ and having coefficients of class $h^{1+\alpha}(\Gamma)$. Hence replacing $\mathbb{M}(\rho)$ of (22) by

$$\mathbb{M}_F(\rho) := F_1(\rho) \cdot \mathbb{M}(\rho) - F_0(\rho) \cdot \mathbb{D}_1(\rho) + P(\rho) \cdot F_1(\rho) \cdot Q(\rho) \cdot \mathbb{D}_{-1}(\rho)$$

we conclude that

$$\mathbb{G}_F(\rho) + \mathbb{M}_F(\rho) \in \mathcal{H}(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)). \tag{24}$$

(iv) Invoking Lemma 3.1 and (18) and we see that

$$\partial\Phi(\rho) - \mathbb{G}_F(\rho) - \mathbb{M}_F(\rho) = F_1(\rho) \cdot \mathbb{L}(\rho)$$

is a rank 1 operator and, thus in particular, compact. Combing (24) and Corollary 3.7 in [3], the proof is completed.

□

We observe that Corollary 3.7 in [3] simply states that additive compact perturbations of generators of analytic C_0-semigroups also generate such a semigroup.

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