Mathematical Analysis of an Electrostatically Actuated MEMS Devices

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ABSTRACT

We perform a rigorous mathematical analysis of a simple membrane based model of an electrostatically actuated MEMS device. Using both analytical and numerical techniques, we prove the existence of a fold in the solution space of the displacement, implying the existence of a critical voltage beyond which there are no solutions of the equation. This critical voltage corresponds to the pull-in voltage observed in simpler lumped models, the numerical solution of three dimensional models, and also in experimental devices. We show how pseudo-arclength continuation may be used to efficiently compute the solutions on both sides of the fold.

Keywords: Actuator, MEMS, Asymptotics, Modeling, Continuation

1 INTRODUCTION

A large number of MEMS devices which rely on electrostatic actuation have been investigated both experimentally [1], [3] and through numerical simulation [2], [3], [5]. However, the mathematical modeling and analysis of this effect has been relatively limited. A simple lumped mass and spring model was first introduced by Nathanson et. al. in 1967 [1]. Subsequently, numerous authors have rediscovered and discussed lumped mass-spring type models, see. e.g., [2]–[4]. Other authors have considered more realistic models, for example, beam theory is employed by Tilmans et. al. in [6], and the full equations of linear elasticity are employed by Funk et. al. in [5]. Nonetheless, the analysis of such models has primarily relied upon linearization or numerical simulation. In this paper, we introduce an idealized model of an electrostatically actuated MEMS device. A summary of the analysis of this model, including asymptotic, rigorous analytical and numerical results, is presented. The detailed analysis is given in [7].

2 THE MODEL

We consider an elastic membrane suspended above a rigid plate. Both membrane and plate are assumed to be infinite in the z' direction, of width L, and separated by a gap of length l, as shown in Figure 1. A potential difference V is applied between the membrane and the plate, which are assumed to be perfect conductors.

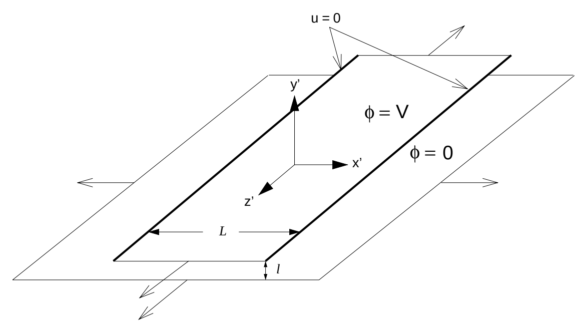

Figure 1: Model of electrostatically actuated device.

Using the dimensionless variables $\psi = \phi/V$, $u = u'/l$, $x = x'/L$, $y = y'/l$, and $t = t'T/\nu L^2$, where u' is the displacement of the membrane, T is the membrane tension, ν is the viscosity, and ϕ is the potential, it can be shown that the governing equations of the system are

$$\epsilon^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \quad (1)$$

$$\frac{1}{\alpha}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -\beta\left(\epsilon^2 (\frac{\partial \psi}{\partial x})^2 + (\frac{\partial \psi}{\partial y})^2\right), \quad (2)$$

with $u(\pm 1/2) = 0$ and $\psi(x, -1) = 0$, $\psi(x, 0) = 1$. Here $\epsilon = l/L$ is the aspect ratio of the device and $\beta = V^2 L^2/8\pi T l^3$ is a dimensionless number which characterizes the ratio between the electrostatic and mechanical forces in the system.

Sending the aspect ratio to zero allows us to explicitly solve for the potential, $\psi = (1+y)/(1+u)$, reducing the problem to one for u alone. If we restrict our attention to steady state solutions this is

$$\frac{d^2 u}{dx^2} = \frac{\beta}{(1+u)^2}. \quad (3)$$

This is the system we shall study throughout the remainder of the paper.

3 ANALYSIS

3.1 Exact Solution

We note that (3) is exactly solvable, yielding an implicit formula for $u(x)$, valid for $\beta \geq 0$:

$$\sqrt{\frac{(u+1)(u+1-\beta/E)}{2E}} + \frac{\beta \tanh^{-1}\sqrt{\frac{u+1-\beta/E}{u+1}}}{E\sqrt{2E}} = x. \tag{4}$$

Here the constant E is readily identified as the total energy (per unit length). Applying $u(1/2) = 0$ we obtain a further implicit formula for the energy as a function of β which can be solved numerically. By careful searching we find two solutions for $0 \leq \beta < \beta^*$. The two values of E correspond to two different solutions $u(x)$. In Figure 2 we plot $u(x=0, \beta)$ for $\beta \geq 0$ and in Figure 3 we show $u(x)$ for various values of β. Now $\beta'(E) = 0$ at the critical value and using this condition and the expression for $\beta(E)$ we can numerically compute the critical value of β to arbitrary precision using any standard mathematical computing package. Doing this, we find that the first 15 digits of β^* are 1.40001647737100.

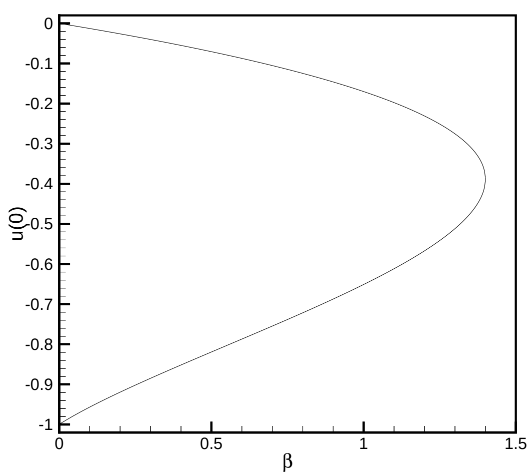

Figure 2: Central displacement of the membrane as a function of β.

3.2 Asymptotic Analysis

In this section, we employ perturbation techniques to construct asymptotic approximations to the solutions uncovered through our study of the exact solution.

We begin with the assumption that $\beta \ll 1$ and first seek approximations to the solutions on the upper part of Figure 3. It can be shown that the first two terms in a power series expansion of the solution for β small are

$$u(x) \sim \beta\left(\frac{x^2}{2} - \frac{1}{8}\right) + \beta^2\left(-\frac{x^4}{4} + \frac{x^2}{8} - \frac{1}{64}\right) + O(\beta^3). \tag{5}$$

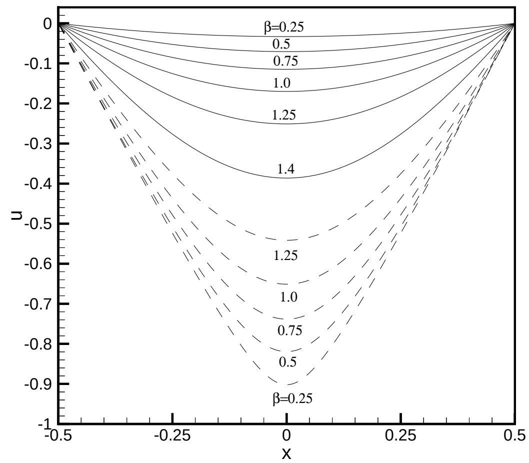

Figure 3: Displacement of the membrane for various values of β.

Now we turn our attention to the goal of uncovering the existence of other solutions to our problem. We rewrite our governing equations in the form

$$(1+u)^2 \frac{d^2u}{dx^2} = \beta \tag{6}$$

and note that for $\beta = 0$, equation (6) posseses the solutions $u(x) = ax + b$ and $u(x) = -1$. If we attempt to impose the boundary conditions $u(\pm 1/2) = 0$, on either solution, we are either faced with a contradiction, or are led to the trivial solution. However, we may attempt to use these linear pieces to build boundary layer solutions. That is, we may allow the equation to be satisfied in various regions by the linear solution $u(x) = ax + b$, and then patch these regions together to obtain a smooth solution by inserting boundary layers where necessary. Clearly, there are multiple possibilities to consider. We may immediately rule out many of them by noting that any solution to the equation must be negative definite, symmetric about the point $x = 0$, and convex (this is shown in the next section). We consider one possible construction, which is to take as an outer solution the V-shaped function:

$$u_{\text{outer}}(x) = c\,|x| - c/2. \tag{7}$$

In order to construct a smooth solution, it is necessary to insert a corner layer at $x = 0$. It can be shown that this is only possible if $c = 2$ and from this we may conclude that only two solutions exist for $\beta > 0$.

3.3 Functional Analysis

In this section we shall use different techniques to prove some additional properties of the model. We shall show that for small negative β a unique positive solution exists to the problem. As β crosses zero a second solution appears. The first upper solution is small and negative and the second lower one is close to the V-shaped function $[x \mapsto 2|x| - 1]$. Most of the results of this section will remain valid for convex domains in higher dimensions and hence with little effort may be applied to more complex models.

We first reformulate (3) as a fixed-point equation for u In this particular situation the Green's function can be computed and we find

$$G(x,\xi) = \begin{cases} (x-\frac{1}{2})(\xi+\frac{1}{2}), & \xi \le x \\ (x+\frac{1}{2})(\xi-\frac{1}{2}), & \xi \ge x \end{cases} \tag{8}$$

which leads to

$$u(x) = F(u,\beta) = \beta \int_{-1/2}^{1/2} \frac{G(x,\xi)}{(1+u(\xi))^2}\, d\xi. \tag{9}$$

From this it can easily be shown that u is an even function of x and that it is convex and that $u(x) \ge u(0)$.

We show in [7] that it is possible to view the solution set $\{(\beta,u) \mid u - F(u,\beta) = 0\}$ as a manifold in the product space $R \times H^2(I) \cap H^1_0(I)$. This manifold can be described as the graph of a function of β in a neighborhood of $(0,0)$ which is trivially a point on the manifold. It is possible to show that for any smooth solution $u \in H^2(I)$ we have $-1 < u(x) < \frac{1}{2}\beta(x+\frac{1}{2})(x-\frac{1}{2})$. Hence there exists $\beta^* > 0$ such that no smooth solution exists with $\beta \ge \beta^*$ and as a first approximation we have $\beta^* \le 8$. By another method we can show the much sharper result $\beta^* \le \frac{4}{27}|\lambda_1| \approx 1.46$ where $|\lambda_1| = \pi^2$ is the principal eigenvalue of the Laplacian.

The main result in this section is contained in a theorem which states that the solution has to decrease as a function of β and that that there is $\beta^* > 0$ for which the solution set is no longer the graph of a function of β. In other words the solution curve $(\beta, u(\beta))$ bends back at $(\beta^*, u(\beta^*))$. This implies that there must be a second solution for $\beta < \beta^*$. From Figure 3 we can see that it looks like as if the lower branch of the solution had some limit as β goes to zero. We believe based on the corner layer analysis that the convergence is towards the V-shaped function $2|x| - 1$, which has finite energy but doesn't lie in the space $H^2(I) \cap H^1_0(I)$. The fact that u converges towards the V-shaped function is no surprise since we know by now that any solution needs to be strictly convex for positive β and that the V-shaped solution is the only nontrivial solution of the limiting equation $u_{xx}(1+u)^2 = 0$ compatible with this property.

4 STABILITY

If we assume that the dynamics of the device are viscosity dominated we may study the stability of the states by adding a $-u_t$ term to the left hand side of (3). Letting $u(x,t) = u_0(x) + v(x,t)$ where u_0 is a stationary state and $v \ll 1 + u_0$, we may derive the eigenvalue equation

$$\frac{d^2V}{dx^2} = V\left(k - \frac{2\beta}{(1+u_0)^3}\right) \tag{10}$$

where $v = V(x)e^{kt}$. Positive (negative) values of k correspond to unstable (stable) stationary states. In Figure 4 we show smallest eigenvalue k as a function of β for $0 \le \beta \le \beta^*$. Note that the values for $k < 0$ were computed using u_0 on the upper branch while the values for $k > 0$ were computed on lower branch. Note also that the positive eigenvalues are shown only for $\beta > 0.8$. As the membrane approaches the V-shaped limiting state this eigenvalue appears to approach infinity.

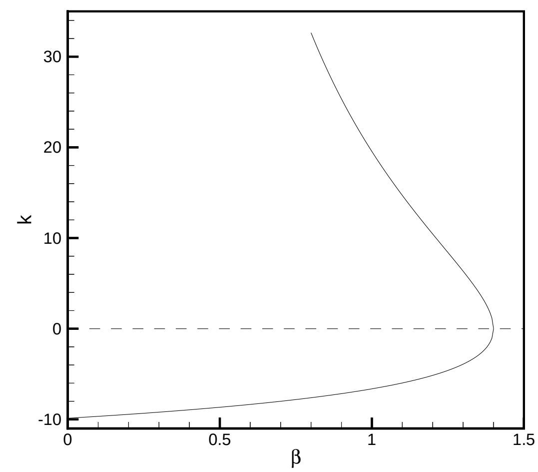

Figure 4: Smallest eigenvalue k as a function of β.

5 ARCLENGTH CONTINUATION

Once it is known that the solution to the problem has a fold we can use one of the path following techniques due to Keller [11] to efficiently compute solutions for both the stable and unstable branches. Here we show how to use one such method, known as pseudo-arclength continuation with the multilevel Newton method of Aluru and White [9], [10] to produce a completely general algorithm for computing all the stationary states.

We assume that we have "black box" routines which solve the structural and electrostatic problems. The input of the structural solver is a pressure, p, derived from the electrostatic solution, while its output is the displacement of the structure u. The input of the electrostatic routine is this displacement, which determines the geometry of the domain, and its output is the pressure. Hence the coupled electrostatic-mechanical system can be written

$$u - S(p) = 0, \quad p - E(u) = 0 \tag{11}$$

where S stands for the inverse of the structural operator (e.g., linear elasticity) and E stands for the inverse of the electrostatic operator. Note that the electrostatic problem for our case reduces to computing the expression $E(u) = \beta(1+u)^{-2}$ and the structural problem reduces to $S(p) = (d^2/dx^2)^{-1}(p)$.

Now, the parameter β is undesirable because $u'(\beta)$ blows up at the fold. What we would like to do is reparameterize the problem using a new parameter, s, in such a way that $u'(s)$ is smooth at the fold. Inspecting Figure 2 we see that $u(0)$ as a function of arc-length of the curve $(u(0,\beta),\beta)$ should be well-behaved as a function of the length along the curve. Following Keller [11] we use a "pseudo-arc-length" parameter, s, and reparameterize the problem so that both u and β are functions of s. To this end, we add

$$\begin{aligned} N &= \theta\left[\dot{u}(s_0)\cdot(u(s)-u(s_0)) + \dot{p}(s_0)\cdot(p(s)-p(s_0))\right] \\ &\quad +(1-\theta)\dot{\beta}(s)(\beta(s)-\beta(s_0)) - (s-s_0) = 0 \quad (12) \end{aligned}$$

to the system (11) making β part of the solution vector. The system for inversion at each Newton step is

$$\begin{pmatrix} I & -\frac{\partial S}{\partial p} & 0 \\ -\frac{\partial E}{\partial u} & I & -\frac{\partial E}{\partial \beta} \\ \theta\dot{u} & \theta\dot{p} & (1-\theta)\dot{\beta} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta p \\ \Delta\beta \end{pmatrix} = - \begin{pmatrix} u^n - S(p^n) \\ p^n - E(u^n) \\ N(u^n, p^n, \beta^n) \end{pmatrix}. \quad (13)$$

Details of how the Jacobian is inverted numerically are given in [7].

In Figure 5 we show $u(0)$ and β as functions of s for $0 \le s \le 8$. Here the range was split into eight steps and the linear approximation (12) was used within each step. Each step was subsequently split into 10 substeps at which the solution was computed. Since the pressure blows up at $x = 0$ as the V-shaped solution is approached the total arc-length of the curve $||u||^2 + ||p||^2$ also blows up. Thus the curves flatten out as s becomes large and $\beta(s)$ will only reach zero in the limit $s \to \infty$.

6 CONCLUSION

We have shown rigorously that a simple membrane based model of an electrostatically actuated MEMS device contains a fold in its solutions space. The fold is responsible for the existence of a critical voltage above which there are no solutions. Additionally, the fold implies the existence of a second solution, which we have computed analytically and numerically. We presented evidence that this second solution is unstable to perturbations while the first solution is stable and showed how pseudo-arclength continuation can be an effective method for computing both the stationary states and the critical voltage. We speculate that these results will apply to more general models in higher dimensions, using elasticity rather than tension, and that actual experimental devices will possess both states, the second state not normally being observable due to its instability.

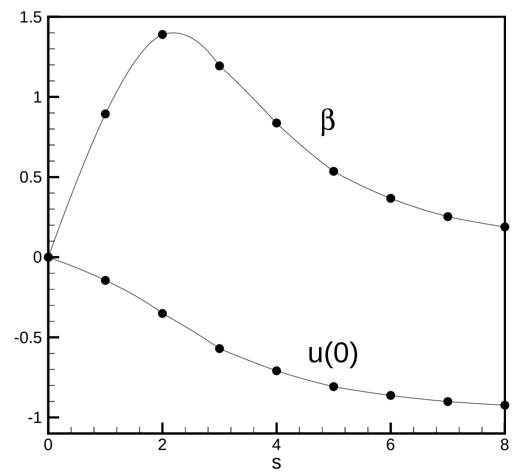

Figure 5: $u(0)$ and β as a function of the arclength parameter s. Solutions to the left of the peak in β are the stable states while those to the right are the unstable states.

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