Midterm Examination I

1. Please indicate with a check mark (✓) which of the equations below are linear (L) and which are nonlinear (N). For the linear ones specify whether they are homogeneous (H) or inhomogeneous (I).

A.

Equation	L	N	H	I
$\tanh(y') + 5y = \cos(t)$		✓		
$e^{3t}y'' + \cos(t)y'' + y^2 = t^5$		✓		
$(2 + \sin(t))y' + e^{\sin(t)}y = t - 1$	✓			✓
$y'' + e^{y'}y = 8$		✓		
$y''' + t^2y'' + e^{-t}y' = \pi y$	✓		✓	

B.

Equation	L	N	H	I
$(3 + e^{t^2+1})y' + \sin(t)y = (t + 1)^2$	✓			✓
$5y' + \tanh(t) = \cos(y)$		✓		
$y''' + e^{y''}y = 7\sin(t)$		✓		
$y'' + \cos(y)y'' + y = t^4$		✓		
$y'' + t^2y' + e^{-t}y''' = -y$	✓		✓	

2. Solve the following initial value problem:

A. $\begin{cases} y' = -\frac{\sin(t)}{y^6} \\ y(0) = 1 \end{cases}$ **B.** $\begin{cases} y' = \frac{\cos(t)}{y^4} \\ y(0) = 2 \end{cases}$

Solution:

In both cases separation of variables is the method of choice here. It gives

$$\textbf{A. } \frac{1}{7}y^7(t) - \frac{1}{7}y^7(0) = \int_0^t y^6(\tau)y'(\tau)\,d\tau = -\int_0^t \sin(\tau)\,d\tau = \cos(t) - 1\,,$$
$$\textbf{B. } \frac{1}{5}y^5(t) - \frac{1}{5}y^5(0) = \int_0^t y^4(\tau)y'(\tau)\,d\tau = \int_0^t \cos(\tau)\,d\tau = \sin(t)$$

and therefore the solutions

$$\textbf{A. } y(t) = \big[7\cos(t) - 6\big]^{1/7}\,,$$
$$\textbf{B. } y(t) = \big[5\sin(t) + 32\big]^{1/5}\,.$$

3. Find the general solution of the following equation:

A. $y' + \frac{1}{1+t}y = t$

B. $(t+2)y' + y = t^2 + 2t$

Solution:

Here it is best to use the integrating factor method. In fact

$$\textbf{A. } \big[(t+1)y\big]' = (t+1)y' + y = (t+1)\big[y' + \frac{y}{t+1}\big] = (t+1)t\,,$$
$$\textbf{B. } \big[(t+2)y\big]' = (t+2)y' + y = t^2 + 2t\,,$$

and then

$$\textbf{A. } (t+1)y(t) - y(0) = \frac{1}{3}t^3 + \frac{1}{2}t^2 \text{ or } y(t) = \frac{1}{1+t}y_0 + \frac{\frac{1}{3}t^3 + \frac{1}{2}t^2}{1+t}\,,$$
$$\textbf{B. } (t+2)y(t) - 2y(0) = \frac{1}{3}t^3 + t^2 \text{ or } y(t) = \frac{2}{t+2}y_0 + \frac{\frac{1}{3}t^3 + t^2}{t+2}\,.$$

4. Solve the following initial value problem:

A. $\begin{cases} 2y'' + 20y' + 50y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$ **B.** $\begin{cases} 3y'' - 24y' + 48y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

Solution:

The characterstic equation is given by

$$\textbf{A.}\ r^2 + 10r + 25 = 0\,,\ \textbf{B.}\ r^2 - 8r + 16 = 0$$

leading to the roots

$$\textbf{A.}\ r_{1,2} = -5\,,\ \textbf{B.}\ r_{1,2} = 4\,,$$

and, therefore to the fundamental solution

$$\textbf{A.}\ y_1(t) = e^{-5t}\,,\ y_2(t) = te^{-5t}\,,\ \textbf{B.}\ y_1(t) = e^{4t}\,,\ y_2(t) = te^{4t}\,,$$

It remains to find the constants c_1 and c_2 such that the initial conditions are satisfied for $c_1y_1 + c_2y_2$, which leads to

$$\textbf{A.}\ \begin{cases} c_1 = 1 \\ -5c_1 + c_2 = 0 \end{cases} \quad \text{and } \textbf{B.}\ \begin{cases} c_1 = 0 \\ 4c_1 + c_2 = 1 \end{cases}$$

Finally this gives

$$\textbf{A.}\ y(t) = e^{-5t} + 5te^{-5t}\,,\ \textbf{B.}\ y(t) = te^{4t}\,.$$

5. Find the general solution of the following equation:

A. $y'' - 3y' + 2y = e^t$

B. $y'' + 4y' + 3y = e^{-t}$

Solution:

First we find two linearly independent solutions of the homogeous equation by looking at the characteristic equation

$$\textbf{A.}\ r^2 - 3r + 2 = (r-2)(r-1) = 0\,,$$
$$\textbf{B.}\ r^2 + 4r + 3 = (r+3)(r+1) = 0\,.$$

it gives the solutions

$$\textbf{A.}\ y_1(t) = e^{2t}\,,\ y_2(t) = e^t\,,\ \textbf{B.}\ y_1(t) = e^{-3t}\,,\ y_2(t) = e^{-t}\,,$$

Judicious guessing is a viable alternative for the computation of a particular solution. The appropriate Ansatz reads

$$\textbf{A.}\ Y(t) = Ate^t\,,\ \textbf{B.}\ Y(t) = Bte^{-t}\,,$$

since the right-hand side is also a solution of the homogeneous equation. A straightforward computation gives

$$\begin{aligned}&\textbf{A.}\ Ate^t[2-3+1] + Ae^t[-3+2] = e^t\,,\\ &\textbf{B.}\ Ate^{-t}[3-4+1] + Ae^{-t}[4-2] = e^{-t}\,,\end{aligned}$$

and leads to the general solution

$$\textbf{A.}\ y(t) = -te^t + c_1e^{2t} + c_2e^t\,,\ \textbf{B.}\ y(t) = \frac{1}{2}te^{-t} + c_1e^{-3t} + c_2e^{-t}\,.$$