Final Examination

1. Find the general solution of the following equation:

A. $y''' - 3y' - 2y = 0$

B. $y''' + 3y'' - 4y = 0$

Solution
Looking for a solution in the form $y(t) = e^{\lambda t}$ we are led to the characteristic equation:

$$\textbf{A. } \lambda^3 - 3\lambda - 2 = 0 \text{ and } \textbf{B. } \lambda^3 + 3\lambda^2 - 4 = 0\,.$$

By inspection we see that the first equation admits the solution $\lambda = -1$ and, similarly, that the second admits the solution $\lambda = 1$. Thus both polynomials can be factored to obtain

$$\textbf{A. } (\lambda+1)(\lambda^2 - \lambda - 2) = (\lambda+1)(\lambda+1)(\lambda-2)$$
$$\text{and } \textbf{B. } (\lambda-1)(\lambda^2 + 4\lambda + 4) = (\lambda-1)(\lambda+2)^2$$

This leads to the solutions

A. $y_1(t) = e^{-t}\,,\ y_2(t) = te^{-t}\,,\ y_3(t) = e^{2t}\,.$

B. $y_1(t) = e^{t}\,,\ y_2(t) = e^{-2t}\,,\ y_3(t) = te^{-2t}\,.$

because of the presence of the double roots.

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2. Solve the following initial value problem:

A. $\begin{cases} y' = e^t y\,, \\ y(0) = 1\,. \end{cases}$ **B.** $\begin{cases} y' = e^{-t} y\,, \\ y(0) = 1\,. \end{cases}$

Solution
The equations can be solved by application of the integrating factor method to give

A. $y(t) = \frac{1}{e} e^{e^t}\,.$

B. $y(t) = ee^{-e^{-t}}$.

3. Find two linearly independent solutions of

A. $y'' - t^3 y = 0$

B. $y'' + t^3 y = 0$

[Explain how you get the two solutions explicitly.]

<u>**Solution**</u>

By a regular power series Ansatz $y(t) = \sum_{n=0}^{\infty} a_n t^n$ we are led to the following equation:

$$\textbf{A.}\ \sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} - \sum_{n=0}^{\infty} a_n t^{n+3} =$$

$$2a_2 + 6a_3 t + 12a_4 t^2 + \sum_{n=3}^{\infty}[(n+1)(n+2)a_{n+2} - a_{n-3}]t^n$$

$$\textbf{B.}\ \sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+3} =$$

$$2a_2 + 6a_3 t + 12a_4 t^2 + \sum_{n=3}^{\infty}[(n+1)(n+2)a_{n+2} + a_{n-3}]t^n$$

We get that in both cases $a_2 = a_3 = a_4 = 0$ and the recurrence relation

$$a_{n+2} = \pm\frac{1}{(n+1)(n+2)}a_{n-3}\,,\ n \geq 3\,,$$

for the remaining coefficents. Setting $a_0 = 1$ and $a_1 = 0$ for $y_1(t)$ and $a_0 = 0$ and $a_1 = 1$ for $y_2(t)$, respectively, and using the above information about the other coefficents produces two linearly independent solutions.

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4. Solve the following equations:

$$\textbf{A.}\ \begin{cases} y'' + \frac{4}{t}y' + \frac{9}{t^2}y = 0\,, \\ y(1) = 0\,, \\ y'(1) = 1\,. \end{cases} \qquad \textbf{B.}\ \begin{cases} y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 0\,, \\ y(1) = 1\,, \\ y'(1) = 0\,. \end{cases}$$

<u>**Solution**</u>

The equation is of Euler type with indicial equation

$$\textbf{A.}\ r^2 + 3r + 9 = 0 \text{ and } \textbf{B.}\ r^2 - 4r + 4 = 0\,.$$

which gives the two linearly independent solutions

$$\textbf{A.}\ y_1(t) = t^{-3/2}\cos\bigl(3\sqrt{3}/2\log(|t|)\bigr)\,, y_2(t) = t^{-3/2}\sin\bigl(3\sqrt{3}/2\log(|t|\bigr)\,,$$
$$\textbf{B.}\ y_2(t) = t^2\,,\ y_2(t) = t^2\log(|t|)\,,$$

of the homogeneous equation. A linear combination of these can be used to satisfy the additional conditions and gives

$$\textbf{A.}\ 0 = c_1y_1(1) + c_2y_2(1) = c_1 \text{ and then } 1 = c_2y_2'(1) = 3\sqrt{3}/2c_2$$

and

$$\textbf{B.}\ 1 = c_1y_1(1) + c_2y_2(1) = c_1 \text{ and then } 0 = y_1'(1) + c_2y_2'(1) = 2 + c_2$$

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5. Verify that $y_1(t)$ is a solution of the given equation and compute a second linearly independent solution for

$$\textbf{A.}\ \begin{cases} y_1(t) = 3t^2 - 1\,, \\ (1-t^2)y'' - 2ty' + 6y = 0\,. \end{cases}$$

$$\textbf{B.}\ \begin{cases} y_1(t) = t + 1\,, \\ (2t+1)y'' - 4(t+1)y' + 4y = 0\,. \end{cases}$$

<u>**Solution**</u>
This is done by reduction of order. It is easily verified that $y_1(t)$ is a solution. The second solution can be therefore seeked in the form $y_2(t) = v(t)y_1(t)$. Plugging this into the equation leads to

$$\begin{aligned}\textbf{A.}\ (1-t^2)\bigl[y_1''(t)v(t) + 2y_1'(t)v'(t) + y_1(t)v''(t)\bigr] + \\ - 2t\bigl[y_1'(t)v(t) + y_1(t)v'(t)\bigr] + 6y_1(t)v(t) \\ = (1-t^2)\bigl[(3t^2-1)v''(t) + 12tv'(t)\bigr] - 2t(3t^2-1)v'(t) = 0\end{aligned}$$
$$\begin{aligned}\textbf{B.}\ (2t+1)\bigl[y_1''(t)v(t) + 2y_1'(t)v'(t) + y_1(t)v''(t)\bigr] + \\ - 4(t+1)\bigl[y_1'(t)v(t) + y_1(t)v'(t)\bigr] + 4y_1(t)v(t) \\ = (2t+1)\bigl[(t+1)v''(t) + 2v'(t)\bigr] - 4(t+1)^2v'(t) = 0\end{aligned}$$

and eventually to:

$$\textbf{A.}\ \frac{v''}{v'} = \frac{2t}{1-t^2} - 2\frac{6t}{3t^2-1}\,,\ \textbf{B.}\ \frac{v''}{v'} = 2 + \frac{2}{2t+1} - 2\frac{1}{t+1}\,.$$

From this we obtain

$$\textbf{A. } v'(t) = \frac{1-t^2}{(3t^2-1)^2}\,, \textbf{B. } v'(t) = e^{2t}\frac{2t+1}{(t+1)^2}\,.$$

This is an acceptable answer.

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6. A. A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, determine its position at time t in the absence of damping. When does the mass return to its equilibrium position for the first time?

B. A mass of 300 g stretches a spring 15 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 5 cm/s, determine its position at time t in the absence of damping. When does the mass return to its equilibrium position for the first time?

Solution

Remember that the spring constant k can be obtained from the equation $k\triangle x = mg$ knowing the displacement $\triangle x$ at equilibrium, the mass m and gravity $g = 10$. In both cases the $k = 20$. Taking into account the given initial conditions, the deviation from equilibrium $y(t)$ satisfies

$$\textbf{A. } \begin{cases} \frac{1}{10}y'' + 20y = 0\,, \\ y(0) = 0\,, \\ y'(0) = \frac{1}{10}\,. \end{cases} \qquad \textbf{B. } \begin{cases} \frac{3}{10}y'' + 20y = 0\,, \\ y(0) = 0\,, \\ y'(0) = \frac{1}{20}\,. \end{cases}$$

The equation has solutions

$$\textbf{A. } y_1(t) = \cos(10\sqrt{2}t)\,,\ y_2(t) = \sin(10\sqrt{2}t)\,,$$
$$\textbf{B. } y_1(t) = \cos(10\sqrt{2/3}t)\,,\ y_2(t) = \sin(10\sqrt{2/3}t)\,.$$

Imposing the intial conditions on the general solution $c_1y_1(t) + c_2y_2(t)$ gives in both cases that $c_1 = 0$ and

$$\textbf{A. } 10\sqrt{2}c_2 = \frac{1}{10}\,, \textbf{B. } 10\sqrt{2/3}c_2 = \frac{1}{20}\,.$$

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7. Find all *singular* points of the given equation and determine whether each one is *regular* or *irregular*.

A. $t^2(1-t)y'' + (t-2)y' - 3ty = 0$

B. $t^2(1-t)^2y'' + 2ty' + 4y = 0$

Solution

In both cases the only singular points are located at $t = 0$ and $t = 1$. Bringing the equations in normal form $y'' + p(t)y' + q(t)y = 0$ we have

$$\textbf{A.}\ tp(t) = \frac{t-2}{t(1-t)}\,,\ t^2q(t) = \frac{3t}{1-t}$$

$$\text{and }\textbf{B.}\ tp(t) = \frac{2}{(1-t)^2}\,,\ t^2q(t) = \frac{4}{(1-t)^2}\,,$$

and thus $t = 0$ is an irregular singular point for **A** and a regular singular point for **B**. As for $t = 1$ we have

$$\textbf{A.}\ (t-1)p(t) = \frac{2-t}{t^2}\,,\ (t-1)^2q(t) = \frac{3(t-1)}{t}$$

$$\text{and }\textbf{B.}\ (t-1)p(t) = -\frac{2}{t(1-t)}\,,\ (t-1)^2q(t) = \frac{4}{t^2}\,,$$

making it a regular singular point for **A** and an irregular one for **B**.

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8. Solve the equation

$$\textbf{A.}\ \begin{cases} y'' + 2y' + 2y = 5\delta(t-\pi)\,, \\ y(0) = 1\,, \\ y'(0) = 0\,. \end{cases} \qquad \textbf{B.}\ \begin{cases} y'' + 4y = 2\delta(t-2\pi)\,, \\ y(0) = 0 \\ y'(0) = 0\,. \end{cases}$$

Solution

By taking a Laplace transform of the equation we obtain

A. $\hat{y}(s) = \frac{s+2}{s^2+2s+2} + \frac{5e^{-\pi s}}{s^2+2s+2}$.

B. $\hat{y}(s) = \frac{2e^{-2\pi s}}{s^2+4}$.

Observing that $\frac{s+2}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$ we arrive at

$$\textbf{A.}\ y(t) = e^{-t}\cos(t) + e^{-t}\sin(t) + 5h_0(t-\pi)e^{-(t-\pi)}\sin(t-\pi)\,.$$

As for the other equation

$$\textbf{B.}\ y(t) = h_0(t-2\pi)\sin(t-2\pi)\,.$$

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9. Solve the initial value problem

A. $y' = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} y\,,\ y(0) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

B. $y' = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} y\,,\ y(0) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

Solution
The system can be rewritten as

$$\textbf{A.}\ \begin{cases} y_1' = y_1 - y_3 \\ y_2' = 2y_2 \\ y_3' = -y_1 + y_3 \end{cases} \qquad \textbf{B.}\ \begin{cases} y_1' = 2y_1 + 2y_3 \\ y_2' = y_2 \\ y_3' = 2y_1 + 2y_3 \end{cases}$$

Clearly the second equation is independent from the others and can be solved by itself to give

$$\textbf{A.}\ y_2(t) = e^{2t}\ \textbf{B.}\ y_2(t) = 2e^t$$

where the initial condition was also taken into account. As for the remaining equations observe that

$$\textbf{A.}\ \big(y_1(t)+y_3(t)\big)' = y_1'(t)+y_3'(t) = 0\ \textbf{B.}\ \big(y_1(t)-y_3(t)\big)' = y_1'(t)-y_3'(t) = 0$$

as follows by adding and subtracting the first and the third equation, respectively. This gives

$$\textbf{A.}\ y_3(t) = c - y_1(t)\ \textbf{B.}\ y_3(t) = y_1(t) - c\,,$$

and the system reduces to the single equation

$$\textbf{A.}\ y_1'(t) = -c + 2y_1(t)\ \textbf{B.}\ y_1(t) = 4y_1(t) - c\,,$$

which can be solved (integrating factor) to yield

$$\textbf{A.}\ y_1(t) = y_1(0)e^{2t} - c\int_0^t e^{2(t-\tau)}\,d\tau = e^{2t} + \frac{c}{2}(1 - e^{2t})$$

$$\textbf{B.}\ y_1(t) = y_1(0)e^{4t} - c\int_0^t e^{4(t-\tau)}\,d\tau = 2e^{4t} + \frac{c}{4}(1 - e^{4t})\,.$$

Thus

$$\textbf{A.}\ y_3(t) = c - y_1(t) = \frac{c}{2}(1 + e^{2t}) - e^{2t}$$
$$\textbf{B.}\ y_3(t) = y_1(t) - c = 2e^{4t} - \frac{c}{4}(3 + e^{4t}),$$

Imposing the remaining initial condition gives

$$\textbf{A.}\ y_3(0) = c - 1 = 2 \ \textbf{B.}\ y_3(0) = 2 - c = 1,$$

and the constant c can be determined. Clearly we could have solved the problem also by computing eigenvalues and eigenvectors of the matrix A just like we learned in class.