Final Examination - Solutions

1. Compute the solution of the following system

$$\begin{cases} \dot{x}_1 = -2x_1 + x_2 + 2t\,, & x_1(0) = 1 \\ \dot{x}_2 = x_1 - 2x_2 - 2t\,, & x_2(0) = 1\,. \end{cases}$$

Solution. The solution of the equation is given by the variation of parameters formula

$$X(t) = e^{tA}X_0 + \int_0^t e^{(t-\tau)A}G(\tau)\,d\tau\,,\ t > 0\,,$$

where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\,,\ X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\,,\ G(t) = \begin{bmatrix} 2t \\ -2t \end{bmatrix} = 2t\begin{bmatrix} 1 \\ -1 \end{bmatrix}\,.$$

In order to compute the exponential of the matrix A we determine its eigenvalues and eigenvectors. The eigenvalues λ are determined through

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - 1 = 0$$

which gives $\lambda_1 = -1$ and $\lambda_2 = -3$. The associated eigenvectors X_λ are then given as solutions of

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} X_{-1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\,,\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} X_{-3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\,.$$

We therefore obtain

$$X_{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\,,\ X_{-3} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\,.$$

The solution formula becomes

$$X(t) = e^{t\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t 2\tau\, e^{(t-\tau)\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\,d\tau =$$

$$e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t 2\tau\, e^{-3(t-\tau)}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\,d\tau = \begin{bmatrix} e^{-t} + \frac{2}{9}e^{-3t} + \frac{2}{3}t - \frac{2}{9} \\ e^{-t} - \frac{2}{9}e^{-3t} - \frac{2}{3}t + \frac{2}{9} \end{bmatrix}\,,\ t > 0\,.$$

---

2. a. Compute the location $t_0 > 0$ where $y(t_0) = 0$ for the function y which solves

$$y' = -\frac{t}{y^2}\,,\ y(0) = y_0 > 0\,.$$

b. Compute $y_\infty = \lim_{t\to\infty} y(t)$ for the solution of

$$y' = (1+y)^2\,e^{-t}\,,\ y(0) = y_0 > 0\,.$$

Solution.

a. The solution can be computed by separating the variables to obtain

$$\int_0^t y'(\tau)y(\tau)^2\,d\tau = \frac{1}{3}y^3(t) - \frac{1}{3}y^3(0) = -\int_0^t \tau\,d\tau = -\frac{1}{2}t^2\,,$$

which gives $y(t) = \big(y_0^3 - \frac{3}{2}t^2\big)^{1/3}$. The zero of interest is therefore $t_0 = \sqrt{\frac{2}{3}y_0^3}$.

b. The solution can again be computed by separation

$$-\int_0^t \frac{y'(\tau)}{\big(1+y(\tau)\big)^2}\,d\tau = \frac{1}{1+y(t)} - \frac{1}{1+y(0)} = \int_0^t e^{-\tau}\,d\tau = 1 - e^{-t}\,,$$

to obtain $y(t) = \frac{1}{\frac{1}{1+y_0}+1-e^{-t}} - 1$ and finally

$$\lim_{t\to\infty} y(t) = \frac{1}{\frac{1}{1+y_0}+1} - 1 = -\frac{1}{2+y_0}\,.$$

---

3. a. Solve

$$\begin{cases} y'' - 2y' + y = 1 + e^{-t}\,, \\ y(0) = 0\,,\ y'(0) = 1. \end{cases}$$

b. Find the solution of

$$\begin{cases} 4y'' + 4y' + 5y = 0\,, \\ y(0) = 1\,,\ y'(0) = 0. \end{cases}$$

Solution

a. The characteristic equation $\lambda^2 - 2\lambda + 1$ has the double root $\lambda = 1$. The general solution of the homogeneous equation is therefore given by

$$y_h(t) = c_1 e^t + c_2 t\, e^t\,.$$

A particular solution y_i of the inhomogeneous equation can be looked for in the form

$$y_i(t) = A + Be^{-t}\,.$$

Plugging this Ansatz into the equation gives

$$Be^{-t} - 2\big(-Be^{-t}\big) + A + Be^{-t} = A + 4Be^{-t} = 1 + e^{-t}\,,$$

from which follows $A = 1$ and $B = 1/4$. Finally the constants c_1 and c_2 have to be determined imposing the initial condition

$$y_h(0) + y_i(0) = c_1 + 1 + 1/4 = 0\,,\ y_h'(0) + y_i'(0) = c_1 + c_2 - 1/4 = 1\,.$$

This gives $c_1 = -5/4$ and $5/2$.

b. The characteristic equation $4\lambda^2 + 4\lambda + 5$ can be solved to obtain $\lambda_{1,2} = -\frac{1}{2} \pm i$ which leads to the general solution

$$y(t) = c_1 e^{-t/2}\cos(t) + c_2 e^{-t/2}\sin(t)\,.$$

The initial condition gives

$$y(0) = c_1 = 1\,,\ y'(0) = -c_1/2 + c_2 = 0 \text{ and finally } c_1 = 1\,,\ c_2 = 1/2\,.$$

---

4. Let $0 < \varepsilon < 1$ and solve

$$\begin{cases} y'' + 2\varepsilon y' + y = \cos(t)\,, \\ y(0) = 0\,,\ y'(0) = 1. \end{cases}$$

What is the amplitude of the solution for large time?

Solution. The characteristic equation has solutions $\lambda_{1,2} = -\varepsilon \pm i\sqrt{1-\varepsilon^2}$ and therefore

$$y_h(t) = c_1 e^{-\varepsilon t}\cos(t\sqrt{1-\varepsilon^2}) + c_2 e^{-\varepsilon t}\sin(t\sqrt{1-\varepsilon^2})\,.$$

We can seek a solution y_i to the inhomogeneous equation in the form

$$y_i(t) = A\cos(t) + B\sin(t)\,.$$

Plugging into the equation

$$\begin{aligned}&-A\cos(t) - \sin(t)B + 2\varepsilon(-A\sin(t) + B\cos(t)) + A\cos(t) + B\sin(t) = \\ &\qquad \big(-A + 2\varepsilon B + A\big)\cos(t) + \big(-B - 2\varepsilon A + B\big)\sin(t) = 2\varepsilon B\cos(t) - 2\varepsilon A\sin(t) = \cos(t)\end{aligned}$$

we see that $A = 0$ and $B = 1/2\varepsilon$. The initial condition then gives

$$y(0) = y_h(0) + y_i(0) = c_1 = 0\,,\ y'(0) = c_2\sqrt{1-\varepsilon^2} + 1/2\varepsilon = 1\,.$$

and the solution is

$$y(t) = \frac{2\varepsilon - 1}{2\varepsilon\sqrt{1-\varepsilon^2}} e^{-\varepsilon t}\sin(t\sqrt{1-\varepsilon^2}) + \sin(t)/2\varepsilon\,.$$

Its asymptotic amplitude is therefore $1/2\varepsilon$.

---

5. Solve the following linear system

$$\begin{cases} \dot{x}_1 = -2x_1 + x_2\,, & x_1(0) = 0 \\ \dot{x}_2 = -2x_2 + x_3\,, & x_2(0) = 1 \\ \dot{x}_3 = -2x_3\,, & x_3(0) = 1 \end{cases}$$

Solution. We need to compute the exponential of the matrix $M = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$. Consider the matrix

$$A(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\,,\ A(-2) = M\,,$$

and recall that

$$e^{tA(\lambda)} = \sum_{k=0}^{\infty} \big(tA(\lambda)\big)^k/k!\,.$$

Computing the powers of $A(\lambda)$ we see that

$$A(\lambda)^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}\,,\ A(\lambda)^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}\,,\ A(\lambda)^4 = \begin{bmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{bmatrix}$$

Plugging into the series we obtain

$$1 + t\lambda + t^2\lambda^2/2 + \cdots + t^k\lambda^k/k! + \cdots = e^{\lambda t}$$

on the diagonal. On the second and third upper diagonals we have

$$0 + t + t^2 2\lambda/2 + t^3 3\lambda^2/3! + \cdots + t^k k\lambda^{k-1}/k! = t\,e^{\lambda t} \text{ and}$$

$$0 + 0 + t^2/2 + t^3 3\lambda/3! + t^4 6\lambda^2/4! + \cdots + t^k(k-1)k\lambda^{k-2}/k! + \cdots = t^2\,e^{\lambda t}/2\,,$$

respectively. We finally obtain

$$X(t) = e^{tA(2)} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & t\,e^{2t} & t^2\,e^{2t}/2 \\ 0 & e^{2t} & t\,e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t\,e^{2t} + t^2\,e^{2t}/2 \\ e^{2t} + t\,e^{2t} \\ e^{2t} \end{bmatrix}$$

---

6. Solve

$$\begin{cases} (1+t^2)y'' + y = 0\,, \\ y(0) = 0\,,\ \ y'(0) = 1. \end{cases}$$

expanding the solution in a power series about $t = 0$ and determine its radius of convergence. Compute only the first five coefficients of the expansion.

Solution. We look for the solution in the form $\sum_{k=0}^{\infty} a_k t^k$. The radius of convergence of the series is given by the distance between $t = 0$ and the closest zero of $t^2 + 1$. Since the zeros are $\pm i$ we obtain a radius of convergence $\rho = 1$. Plugging the series Ansatz into the equation we obtain

$$(1+t^2)\sum_{k=2}^{\infty} a_k k(k-1)t^{k-2} + \sum_{k=0}^{\infty} a_k t^k =$$

$$a_0 + 2a_2 + t[6a_3 + a_1]t + \sum_{k=2}^{\infty}\big[(k+1)(k+2)a_{k+2} + (k^2-k+1)a_k\big]t^k$$

The initial condition entails $a_0 = 0$ and $a_1 = 1$. The remaining coefficients can be obtained via the recurrence relation

$$a_{k+2} = -\frac{k^2-k+1}{(k+1)(k+2)}a_k\,,\ k = 0, 1, 2, \ldots$$

to give $a_2 = 0$, $a_3 = -1/6$, $a_4 = 0$.

---

7. Consider the following equations

$$(x^2-1)y'' + \sin((x-1)\pi/4)y' + (x^2-3x+2)y = 0\,,$$

$$(x^3-1)y'' + 2xy' - y = 0\,,$$

$$xy'' + \sin(\frac{1}{x})y' + 2y = 0\,.$$

a. Determine their singular points and whether they are regular?
b. For the first equation determine the exponents at the singularity.
c. What can you say about the radius of convergence of a series solution of the second equation about $x = 1$.

Solution
a. The point $x = -1$ is a regular singular point for the first equation, $x = 1$ a regular singular point of the second whereas $x = 0$ is not a regular singular point for the last equation.

b. The exponents at the singularity for the first equation are the solutions of the inditial equation $r(r-1)+p_0r+q_0r^2=0$ where

$$p_0=\lim_{x\to -1}(x+1)\frac{\sin((x-1)\pi/4)}{(x+1)(x-1)}=1/2\,,\; q_0=\lim_{x\to -1}(x+1)^2\frac{(x-2)(x-1)}{(x+1)(x-1)}=0\,.$$

We therefore obtain $r_{1,2}=0,1/2$.

c. The other solutions of $x^3-1=(x^2+x+1)(x-1)=0$ are given by

$$x_{1,2}=\frac{1}{2}\big(-1\pm\sqrt{1-4}\big)=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\,.$$

The distance to the closest zero from $x=1$ is given by $\sqrt{3}$. The radius of convergence is therefore at least $\sqrt{3}$.

---

8. A bungee jumper jumps off a $180m$ high bridge attached to a $125m$ long cord. His body mass is $m=80kg$ and the cord behaves like a spring with constant $k=160kg/sec^2$. How close to the ground does the jumper come? Neglect air resistance and assume $g=10m/sec^2$.

Solution. The equation which determines the displacement from the equilibrium is given by

$$m\,y''+k\,y=80\,y''+160\,y=0\,.$$

To determine the initial conditions we need to find the speed of the jumper when the cord is fully elongated. Using $g=10\,m/sec^2$ we see that the jumper has fallen $125\,m$ after $t=5\,sec$ as can computed from

$$\frac{1}{2}10t^2=125\,.$$

This gives a velocity

$$y'(0)=50\;[m/sec]\,.$$

Since the spring constant is $k=160\,kg/sec^2$ and $g=10\,m/sec^2$ we see that the elongation of the cord at equilibrium is given by $\frac{80\cdot 10}{160}\frac{kg\,m/sec^2}{kg/sec^2}=5\,m$. We finally obtain the initial condition

$$y(0)=-5\;[m]\,.$$

Finally we obtain the equation

$$y''+2y=0\,,\; y(0)=-5\,,\; y'(0)=50\,.$$

Since the general solution is given by

$$c_1\cos(t\sqrt{2})+c_2\sin(t\sqrt{2})$$

we obtain

$$c_1=-5 \text{ and } c_2\sqrt{2}=50\,,$$

which gives an amplitude of $R=\sqrt{25+2*25^2}=5\sqrt{51}\approx 35$. The jumper therefore falls a total of about $165\,m$.

---

9. Solve

$$\begin{cases}\dot{x}_1=-x_1-2x_2\,, & x_1(0)=1\\ \dot{x}_2=2x_1-x_2\,, & x_2(0)=1\,.\end{cases}$$

Solution. The characteristic equation of $A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$ is given by $(-1-\lambda)^2 + 4 = 0$ which gives the eigenvalues

$$\lambda_{1,2} = -1 \pm 2i\,.$$

The associated eigenvectors are given by

$$X_{-1+2i} = \begin{bmatrix} i \\ 1 \end{bmatrix}\,,\; X_{-1-2i} = \begin{bmatrix} i \\ -1 \end{bmatrix}\,.$$

We can use them to obtain

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2i}\big(X_{-1+2i} + X_{-1-2i}\big) \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\big(X_{-1+2i} - X_{-1-2i}\big)\,.$$

Since the solution is given by

$$e^{tA}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{tA}\Big(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\Big)$$

we compute

$$e^{tA}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2i}e^{tA}\big(X_{-1+2i} + X_{-1-2i}\big) = \frac{1}{2i}\big(e^{-t+2it}X_{-1+2i} + e^{-t-2it}X_{-1-2i}\big) = e^{-t}\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$$

$$e^{tA}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}e^{tA}\big(X_{-1+2i} - X_{-1-2i}\big) = \frac{1}{2}\big(e^{-t+2it}X_{-1+2i} - e^{-t-2it}X_{-1-2i}\big) = e^{-t}\begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}$$

which gives the solution

$$e^{tA}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-t}\begin{bmatrix} \cos(2t) - \sin(2t) \\ \cos(2t) + \sin(2t) \end{bmatrix}$$

---

10. Determine the indicial equation and the corresponding exponents at the singularity for the regular singular points of

$$x^2y'' + x^2y' - y = 0$$
$$x\sinh(x)y'' + xy' - y = 0$$
$$\tanh(x)y'' - y' = 0$$

Solution. The point $x = 0$ is the only regular singular point for each equation. The indicial equations and the corresponding exponents are given by

$$r(r-1) - 1 = 0\,,\; r_{1,2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}\,,$$
$$r(r-1) + r - 1 = 0\,,\; r_{1,2} = \pm 1\,,$$
$$r(r-1) + r = 0\,,\; r_{1,2} = 0\,.$$