You Ask a Question Week 7
Math 2A – Winter 2012

This past week we have learned about a few important mathematical theorems. What is a *theorem*? Just like in algebra, when after practicing additions, multiplications and such, you start realizing that there are certain rules which numbers and operations follow (a+b=b+a, ...), in the same way you can start realizing that there are classes of functions which also do follow certain rules or share some common properties. Theorems capture precisely such rule or properties. Since the claims of a theorem, which are nothing but the statement of certain properties, are not typically not valid for any randomly chosen function, a theorem will also have some *hypotheses* which aim at identifying those functions for which their claim do apply. Let us take *Fermat's* theorem as an example. It claims that $f'(x_0) = 0$ for any function $f : (a, b) \to \mathbb{R}$ which has a local extremum (maximum or minimum) at the point $x_0 \in (a, b)$ provided f is differentiable there. Clearly such a claim can only be valid if the function f is differentiable there, thus the assumption. There, are, however, other more subtle assumptions in the statement of the theorem. The fact that we consider a function defined on an open interval is also crucial! Simply by taking the function $f : [0, 1) \to \mathbb{R}\,,\ x \mapsto x$, or $f(x) = x$, you can easily convince yourself that f has a local (actually even global) minimum at $x_0 = 0$. Since $f' \equiv 1$, it is certainly not true that $f'(x_0) = f'(0) = 0$! The function does NOT satisfy the hypotheses of the theorem: It is considered on an interval which contains one of its endpoints! Thus the requirement that $f : (a, b) \to \mathbb{R}$ means that the theorem really only applies to functions considered on an open interval (open interval = interval not containing its endpoints). Let us consider a non-mathematical example. Consider the claim: planes can fly. Clearly the claim is true for planes (and might on occasion be true a bit more in general - birds fly, too) but you can't possibly apply the same conclusion to cars! In the above mathematical example, functions defined on an open interval play the role of planes, while functions considered on a closed interval (closed interval = interval containing its endpoints) play that of cars. What applies to the one category does not necessarily apply to the other! All other theorems we have seen in class come with assumptions, which determine the category of functions to which they apply, and claims, i.e., once more, the statement of one or more properties common to all functions in that category.

Let us take a closer look at Rolle's and at the mean value theorems. Both theorems are about a category of functions. Which functions exactly?

Rolle's theorem: *Functions* $f : [a, b] \to \mathbb{R}$ *which are continuous on the closed interval* $[a, b]$ *and differentiable in the open interval* (a, b) *and which happen to satisfy* $f(a) = f(b)$*, i.e. which take the same value at endpoints of the interval.*
Mean Value theorem (MVT): *Functions* $f : [a, b] \to \mathbb{R}$ *which are continuous on the closed interval* $[a, b]$ *and differentiable in the open interval* (a, b)*.*
We observe that the two categories of functions determined by the above properties are slightly different. While Rolle's theorem only applies with functions attaining the same value at the endpoints of their interval of definition, the mean value theorem does not include this requirement. The remaining assumptions about continuity and differentiability do, however, coincide. It follow, for instance, that the hypotheses of both theorems are satisfied for the function $f(x) = 1 - x^2$ on $[-1, 1]$, while the function $f(x) = 1 - x^2$ on $[0, 2]$ only satisfies the hypotheses of the MVT.
Let us now turn to the claims of these theorems.
Rolle's theorem: *There exists a point* $x_0 \in (a, b)$ *such that* $f'(x_0) = 0$*.*
Mean Value theorem (MVT): *There exists a point* $x_0 \in (a, b)$ *such that* $f'(x_0) = \frac{f(b)-f(a)}{b-a}$*.*
The two claims seem different at least at first sight. If you take the hypotheses of Rolle's theorem, however, we notice that

$$\frac{f(b) - f(a)}{b - a} = 0\,,$$

for all functions which satisfy those assumptions! It follows that the claim is therefore the same for both theorems! The only difference is that the first applies to a smaller class of functions than the second. In more mathematical terms this means that Rolle's theorem is a special case of the MVT, indeed the one for which $f(b)$ happen to coincide with $f(a)$. So why on earth have we learned them both separately? The only reason, as far as I am concerned, is that this made the proofs a little easier. I won't say anything about the geometric interpretation of these theorems since I have devoted time to it in class.

We spent the rest of the week learning about what can be said about the behavior of function by looking at their first and second derivatives. In particular we learned that the first derivative tells us if a function is going up (*increases*) or down (*decreases*) depending on the sign of its derivative. The zeros of the derivative are the points where the derivative (possibly) changes sign and thus help determine the intervals on which the function is increasing or decreasing. The exact claim for differentiable functions is

$$f'(x) > 0 \, \forall \, x \in (a,b) \Longrightarrow \; f \text{ is increasing on } (a,b)\,.$$

or, correspondingly,

$$f'(x) < 0 \, \forall \, x \in (a,b) \Longrightarrow \; f \text{ is decreasing on } (a,b)\,.$$

Finally we turned to the second derivative of a function, which we determines its concavity properties, i.e. whether the function smiles up or down. If you trace out a smile on a piece of paper you will notice that the slope of its tangent line starts out negative but keeps growing and eventually turns positive. If f is the function describing the smile, then, this simply means that f' is increasing. It should not come as a surprise then that

$$f''(x) > 0 \, \forall \, x \in (a,b) \Longrightarrow \; f \text{ is smiling (concave up) on } (a,b)\,.$$

In fact, the preceding considerations tell us that

$$f''(x) > 0 \, \forall \, x \in (a,b) \Longrightarrow \; f' \text{ is increasing on } (a,b)\,,$$

since f'' is nothing but the first derivative of f'!

I remind you that we will use Wednesday's class for a review of the material covered in the next midterm. Come with questions prepared!