You Ask a Question Week 3
Math 2A – Winter 2012

After learning the concept of continuity, we learned the so-called **intermediate VALUE theorem**. As its name indicates it is concerned with function **values**. Recall that, in an expression $y = f(x)$, x is called the *argument* and y the *value* of the function f. Thus the theorem is concerned with the y!
Let a function f be continuous on the interval $[a, b]$ and assume that the values $f(a)$ and $f(b)$ do not coincide. Let m and M be the smallest and the largest of these two value, respectively

$$m = \min\{f(a), f(b)\}\,,\ M = \max\{f(a), f(b)\}\,.$$

The theorem claims that, if the values of f go from $f(a)$ to $f(b)$ while the argument x travels from a to be b, then f has to visit any value y_0 between m and M (i.e. any $y_0 \in [m, M]$) at least one time! In still other words: how could you go from $A = f(3)$ to $B = f(5)$ between $a = 3$PM and $b = 5$PM without passing through $C = y_0$ at some time $t = x_0$ past 3PM and before 5PM, if C is on the way? Visiting a value y means mathematically that an argument x_0 can be found between a and b such that $y_0 = f(x_0)$.
The assumption on continuity is clearly crucial: if you were teleported (see Star Trek) from A to B then you would not necessarily be passing through C but teleportation is not continuous and thus not under the purview of the theorem! This theorem can be used to check whether a given function assumes a given value of interest. This is the case for instance with the value zero for a given polynomial $p(x)$, where the problem simply amounts to trying and solving the equation $p(x) = 0$ by locating arguments a and b where the polynomial is positive and negative, respectively. Notice that the value 0 is (intermediate) between something negative and something positive. Finding such locations is arguably much easier than spotting the zero itself! See your class notes or the textbook for examples of this.

Differentiability is the central concept of this class. As I tried to make clear all along, it is related to the concept of limit. Take a real real-valued function f and recall that, given two points x_0 and x (arguments), the secant line going through them is given by (why? Convince yourself!)

$$y = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0)\,.$$

It gives us some very partial information about the behavior of the function f between the two given arguments. In particular, its slope measures the *average change* of f on the interval between x_0 and x. By making this interval smaller and smaller (moving x closer and closer to x_0 for instance), we get the average change (rate of change, ...) of the function f in a shorter and shorter interval about x_0. It should make intuitive sense that we would get the *instantaneous rate of change* of f in the limit as the interval shrinks to the point x_0. The mathematical concept of limit captures this very reasoning and thus

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

ought to represent such instantaneous rate of change whereas the line

$$y = f(x_0) + f'(x_0)(x - x_0)\,,$$

would have to be the line tangent to the graph of f at $\big(x_0, f(x_0)\big)$. Observe that the limit involved in the computation of the derivative $f'(x_0)$ of f at x_0 is of type $\frac{0}{0}$. We know by now that such limits do need to be taken with care and might *not* exist at all! Again, while we would love to have $f'(x_0)$, the limit involved might not exist; we are therefore forced to make the following *definition*: f is called *differentiable* at x_0 if the limit in (1) exists.
To convince ourselves that the limit does not always exist, we consider an example. Let $f = |x|$ and let us try to compute the limit of the difference quotient at $x_0 = 0$. Let us take a look at the quotient first:

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{|x| - |x_0|}{x - x_0} = \frac{|x| - |0|}{x - 0} = \frac{|x|}{x}\,.$$

Does it remind you of anything? In order to compute the limit as x tends to $x_0 = 0$, we consider x coming from the left and from the right separately because

$$|x| = \begin{cases} x\,, & x \geq 0\,, \\ -x\,, & x < 0\,, \end{cases}$$

is a piecewise defined function. We obtain

$$\lim_{x \to x_0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-x}{x} = \lim_{x \to 0-} -1 = -1\,,$$

and

$$\lim_{x \to x_0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = \lim_{x \to 0+} 1 = 1\,.$$

We now remind ourselves of the fact that a limit $\lim_{x\to x_0} g(x)$ exists if and only if the two one-sided limits $\lim_{x\to x_0\pm} g(x)$ exist and coincide. In our case $g(x) = \frac{|x|}{x}$, $x_0 = 0$, and the two one sided limits exist. They do, however, not coincide. We conclude that the limit does not exist and that the function f is not differentiable at $x_0 = 0$.
Let us obtain a derivative next. Consider the function $f(x) = 1/x$ for $x > 0$ and try and determine its derivative at a generic point $x_0 > 0$. Let us start with the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{1/x - 1/x_0}{x - x_0} = \frac{x_0 - x}{x_0 x} \frac{1}{x - x_0} = -\frac{x - x_0}{x_0 x} \frac{1}{x - x_0} = -\frac{1}{x_0 x} .$$

It only remains to take the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \left(-\frac{1}{x_0 x}\right) = -\frac{1}{x_0} \lim_{x\to x_0} \frac{1}{x} = -\frac{1}{x_0^2} .$$

We just obtained that

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = -\frac{1}{x_0^2}$$

for $x_0 > 0$.