

Assignment 14

1. Assume that $f \in C([0, 1])$ and that

$$F \in C([0, 1] \times [0, 1] \times \mathbb{R}), \quad \partial_u F \in C([0, 1] \times [0, 1] \times \mathbb{R})$$

and consider the integral equation

$$u(x) = \int_0^1 F(x, y, u(y)) dy + f(x), \quad 0 \leq x \leq 1.$$

Show that, if $\|\partial_u F\|_\infty < 1$, the integral equation has a unique solution $u \in C([0, 1])$.

2. Let $f \in C(\mathbb{B}(0, 1), \mathbb{R}^n)$ be such that

$$|f(x)| \leq 1 \text{ for } |x| = 1.$$

Show that f has a fixed point in $\mathbb{B}(0, 1)$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. Prove that the equation

$$\begin{cases} -\Delta u = e^{-u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

possesses a solution.

4. (*Kolmogoroff*) Show that a subset $K \subset L_p(\mathbb{R}^n)$ ($1 \leq p < \infty$) is compact iff

(i) K is closed and bounded.

(ii) $\int_{|x| \geq N} |f(x)|^p dx \rightarrow 0$, $N \rightarrow \infty$, uniformly in $f \in K$.

(iii) $\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx \rightarrow 0$, $|h| \rightarrow 0$, uniformly in $f \in K$.

[Hint: Use the density of test functions in $L_p(\mathbb{R}^n)$, the strong continuity of the translation semigroup on $L_p(\mathbb{R}^n)$ and Arzela-Ascoli.]

5. Let $1 \leq p < \infty$ and prove that

$$\left(\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p dx \right)^{1/p} \leq |h| \|u\|_{1,p}$$

for $u \in W_p^1(\mathbb{R}^n)$. Use this estimate and Kolmogoroff's characterization of compactness to show that $W_p^1(\Omega) \hookrightarrow L_p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open and bounded.

Homework due by Friday, June 1 2007.