Assignment 10

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1. Let $H(t,x)$ be the heat kernel (the fundamental solution for the heat equation introduced in class) and show that T defined through

$$\begin{cases} \big(T(t)u\big)(x) & := \int_{\mathbb{R}^n} H(t,x-y)u(y)\,dy\,,\ x\in\mathbb{R}^n \\ T(0)u & := u \end{cases}$$

is a C_0-semigroup of contractions on $\mathrm{L}_2(\mathbb{R}^n)$ but NOT on $\mathrm{L}_\infty(\mathbb{R}^n)$.

A C_0-semigroup T on a Banach space E is called *analytic* if it allows for an analytic strongly continuous extension to a sector $\Sigma_\delta = [\arg(z) < \delta]$ of the complex plane for some $\delta \in (0,\pi/2]$, that is, if

(i) $T(0) = \mathrm{id}_E\,,\ T(z_1+z_2) = T(z_1)T(z_2)\,,\ z_1,z_2 \in \Sigma_\delta\,.$

(ii) $T : \Sigma_\delta \to \mathcal{L}(E)$ is analytic.

(iii) $\lim\limits_{\Sigma_\delta \ni z \to 0} T(z)x = x$ for all $x \in E\,.$

It can be shown that the above conditions are equivalent to

(i) $T(t)E \subset \mathrm{dom}(A)\,,\ t>0\,.$

(ii) $\|tAT(t)\|_{\mathcal{L}(E)} \le c < \infty\,,\ t>0\,.$

where $-A : \mathrm{dom}(A) \subset E \longrightarrow E$ is the generator of T. Show that the C_0-semigroup of problem 1 is analytic.

2. Let $-A : \mathrm{dom}(A) \subset E \longrightarrow E$ be the generator of an analytic C_0-semigroup T on E. Let $f \in \mathrm{C}^\rho\big([0,T],E\big)$ for some $\rho \in (0,1)$ and show that the mild solution $u : [0,T] \to E$ of

$$\dot u + Au = f(t)\,,\ u(0) = x \in E\,,$$

given by

$$u(t) = T(t)x + \int_0^t T(t-\tau)f(\tau)\,d\tau\,,\ t\in[0,T]$$

is actually differentiable for $t>0$.

3. Let $A : \mathrm{dom}(A) \subset E \longrightarrow E$ be defined through

$$\begin{aligned} &E = \mathrm{L}_2(0,1)\,,\\ &\mathrm{dom}(A) = \big\{u \in \mathrm{H}^2(0,1)\,\big|\, u(0)=u(1)=0\big\}\,,\\ &Au = -\partial_{xx}u\,,\ u\in\mathrm{dom}(A)\,, \end{aligned}$$

and show that $-A$ generates an analytic C_0-semigroup on E.

4. Define $T(t) \in \mathcal{L}\big(\mathrm{L}_2(\mathbb{R}^n)\big)$ through
$$\begin{cases} (1-\triangle)^{-t} = \mathcal{F}^{-1}(1+|\xi|^2)^{-t}\mathcal{F}\,, & t>0 \\ \mathrm{id}_{\mathrm{L}_2(\mathbb{R}^n)}\,, & t=0\,. \end{cases}$$
Show that T is a C_0-semigroup on $\mathrm{L}_2(\mathbb{R}^n)$. What is its generator?

5. For a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, for $b \in \mathrm{L}_\infty(\Omega)$ and $c \in \mathrm{L}_\infty(\Omega)$ let A be the operator induced by the Dirichlet form
$$a(u,v) = \int_\Omega \big[(\nabla u\,|\,\nabla v) + (b\,|\,\nabla u)v + cuv\big]\,dx\,,\ u,v \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$$
on $\mathrm{H}^{-1}(\Omega)$. Show that it generates a C_0-semigroup on $H^{-1}(\Omega)$.

The Homework is due Wednesday, March 14 2007