Assignment 8

1. Let T be a c_0-semigroup on a Banach space E. Prove that there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that
$$\|T(t)\|_{\mathcal{L}(E)} \leq M\, e^{\omega t}\,.$$

2. For a c_0-semigroup T on a Banach space E define
$$\xi(t,x) = T(t)x\,,\ t \in [0,\infty)\,,\ x \in E$$
and prove that the following are equivalent:
(i) $\xi(\cdot,x)$ is differentiable.
(ii) $\xi(\cdot,x)$ is right differentiable.

3. Show that the translation semigroup T on $\mathrm{BUC}(\mathbb{R})$ defined through
$$T(t)f(\cdot) = f(\cdot - t)\,,\ f \in \mathrm{BUC}(\mathbb{R})$$
is strongly continuous and compute its generator.

4. Let $A \in \mathcal{G}(E)$, $x \in E$ and $f \in \mathrm{C}^{1-}\big([0,\infty) \times E, E\big)$ and prove that
$$\begin{cases} \dot{u} + Au = f(t,u)\,,\ t > 0 \\ \quad u(0) = x \end{cases}$$
has a unique local mild solution $u(\cdot,x) \in \mathrm{C}\big([0,t^+(x)),E\big)$ for some $t^+(x) > 0$.
[Hint: Use Banach Fixed-Point Theorem combined with the Variation-of-Constants-Formula]

5. Let $A \in \mathbb{C}^{n\times n}$ and show that
$$e^{-tA} = \frac{1}{2\pi i}\int_{\partial\mathbb{B}(0,R)} e^{\lambda t}(\lambda + A)^{-1}\,d\lambda\,,$$
where $R > 0$ is such that $\sigma(-A) \subset \mathbb{B}(0,R)$ and the integration is counterclockwise.
[Hint: Show that both sides of the equation satisfy the same ODE and prove that they have the same initial value by reducing the problem to the case where A is a Jordan block]

The Homework is due February 18 2005