Assignment 7

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1. Let $\mathcal{A}$ be the general elliptic second order differential operator in divergence from on a bounded domain Ω with smooth boundary, that is,
$$\mathcal{A}u = -\nabla \cdot (A\nabla u) + (b|\nabla u) + cu$$
where the coefficents satisfy the assumptions of Chapter 8 of class. Let $\Phi \in \mathrm{Diff}^2(\Omega, \widetilde{\Omega})$, that is, Φ is invertible and
$$\Phi \in \mathrm{C}^2(\Omega, \widetilde{\Omega}) \,,\ \Psi := \Phi^{-1} \in \mathrm{C}^2(\widetilde{\Omega}, \Omega) \,.$$
Letting $y := \Phi(x)$ and define $\tilde{u}(y) := u(\Psi(y))$, compute the operator $\tilde{\mathcal{A}}$ in the new variables, that is the operator satisfying
$$\widetilde{\mathcal{A}u} = \tilde{\mathcal{A}}\tilde{u} \,.$$
2. Let $f : \Omega \to \mathbb{R}$ be measurable. Define
$$\mu_f(t) := \big|\{x \in \Omega \,:\, |f(x)| > t\}\big|$$
Let $p > 0$ and assume $f \in \mathrm{L}_p(\Omega)$. Prove that
$$\mu_f(t) \leq t^{-p} \, \|f\|_p^p$$
and that
$$\|f\|_p^p = p \int_0^\infty t^{p-1} \mu_f(t) \, dt \,.$$
3. Let $1 \leq q < r < \infty$ and $T : \mathrm{L}_q(\Omega) \cap \mathrm{L}_r(\Omega) \to \mathrm{L}_q(\Omega) \cap \mathrm{L}_r(\Omega)$ be a linear operator such that
$$u_{Tf}(t) \leq \big(T_1\|f\|_q/t\big)^q \text{ and } u_{Tf}(t) \leq \big(T_2\|f\|_r/t\big)^q$$
for some constants T_1 and T_2. Then T can be extended to an operator $T \in \mathcal{L}\big(\mathrm{L}_p(\Omega)\big)$ for any $p \in (q, r)$ and
$$\|Tf\| \leq cT_1^\alpha T_2^{1-\alpha} \|f\|_p \,,\ f \in \mathrm{L}_q(\Omega) \cap \mathrm{L}_r(\Omega)$$
where $1/p = (1-\alpha)/r + \alpha/q$.
[Hint: For $s > 0$ use $f = f\chi_{[|f|>s]} + f\chi_{[|f|\leq s]} = f_1 + f_2$ to prove that
$$\mu_{Tf}(t) \leq \mu_{Tf_1}(t/2) + \mu_{Tf_2}(t/2)$$
and then use the previous problem and Fubini's theorem.]
4. Prove that the Neumann problem
$$\begin{cases} -\triangle u &= f \text{ in } \Omega \,, \\ \partial_\nu u &= 0 \text{ on } \partial\Omega \end{cases}$$
on a bounded domain with smooth boundary has a solution if and only if $\int_\Omega f = 0$.

5. Let $\mathbb{H}^n$ be the upper half-space. Given $m \in \mathbb{N}$ construct an extension operator $\mathrm{ext} : \mathrm{C}^m(\overline{\mathbb{H}^n}) \to \mathrm{C}^m(\mathbb{R}^n)$ such that
$$(\mathrm{ext}\, u)\big|_{\mathbb{H}^n} = u\,,\ u \in \mathrm{C}^m(\overline{\mathbb{H}^n})$$

Homework due by Friday, February 4