Assignment 1

1. Show that $v.p.\frac{1}{x}$ is a well-defined distribution. Recall that

$$\langle v.p.\frac{1}{x}, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} \, dx$$

   for any test function $\varphi \in \mathcal{D}(\mathbb{R})$. Then prove that

$$u_\varepsilon^\pm \to \mp i\pi\delta + v.p.\frac{1}{x} \text{ as } \varepsilon \downarrow 0$$

   for $u_\varepsilon^\pm(x) := \frac{1}{x \pm i\varepsilon}$, $x \in \mathbb{R}$, in the sense of distributions.

2. Compute f' and f'' for $f(x) = \log|x|$, $x \in \mathbb{R}$ and $f(x) = |x|$, $x \in \mathbb{R}$, respectively, in the sense of distributions.

3. Let $\rho : \mathbb{R}^n \mapsto \mathbb{R}$ be an integrable function with

$$\text{supp}(\rho) \subset \mathbb{B}(0,1) \text{ and } \int_{\mathbb{R}^n} \rho(x) \, dx = 1 \,.$$

   Show that $\rho_\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$ for

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho(\frac{x}{\varepsilon}), \ x \in \mathbb{R}^n \,.$$

4. Consider the function u defined as

$$u(x,y) = \frac{1}{\pi} \frac{y}{x^2 + y^2} \,, \ (x,y) \in \mathbb{R} \times (0,\infty)$$

   Then $u \in \mathrm{C}^\infty(\mathbb{R} \times (0,\infty))$. Compute $\lim_{y \to 0} u(\cdot, y)$.

5. Let $\Omega \subset \mathbb{R}^n$ be an open domain with C^1 boundary $\partial\Omega$ and denote by χ_Ω its characteristic function. Compute $\nabla\chi_\Omega$ in the sense of distributions and determine its support.

Homework due by Friday, October 8