Math 295a Fall Term 2001

Assignment 1

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1. Show that $v.p.\frac{1}{x}$ is a well-defined distribution. Recall that
$$\langle v.p.\frac{1}{x}, \varphi\rangle = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,dx$$
for any test function $\varphi\in\mathcal{D}(\mathbb{R})$. Then prove that
$$u_\varepsilon^\pm \to \mp i\pi\delta + v.p.\frac{1}{x} \text{ as } \varepsilon\downarrow 0$$
for $u_\varepsilon^\pm(x) := \frac{1}{x\pm i\varepsilon}$, $x\in\mathbb{R}$, in the sense of distributions.

2. Compute f' and f'' for $f(x)=\log|x|$, $x\in\mathbb{R}$ and $f(x)=|x|$, $x\in\mathbb{R}$, respectively, in the sense of distributions.

3. Let $\rho:\mathbb{R}\mapsto\mathbb{R}$ be an integrable function with
$$\operatorname{supp}(\rho)\subset[0,1] \text{ and } \int_{-\infty}^{\infty}\rho(x)\,dx = 1\,.$$
Show that $\rho_\varepsilon\to\delta$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon\to 0$ for $\rho_\varepsilon(x) := \frac{1}{\varepsilon}\rho(\varepsilon x)$, $x\in\mathbb{R}$.

4. Consider the function u defined as
$$u(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}\,,\ (x,y)\in\mathbb{R}\times(0,\infty)$$
Then $u\in \mathrm{C}^\infty(\mathbb{R}\times(0,\infty))$. Compute $\lim_{y\to 0} u(\cdot,y)$.

5. Assume that $f\in\mathrm{C}^1(\mathbb{R}^n,\mathbb{R})$, then
$$T_{\partial^j f} = \partial^j T_f\,,\ j=1,\dots,n$$
that is, the derivative in the sense of distributions coincides with the classical derivative if the latter exists. Let now f be continuously differentiable except at finitely many points $x_1,\dots,x_m$ $(m\in\mathbb{N})$ where
$$\lim_{x\downarrow x_j} f(x) - \lim_{x\uparrow x_j} f(x) = \sigma_j\,,\ j=1,\dots,m\,.$$
Compute its derivative in the sense of distributions.