Assignment 2

A linear operator $A : \mathrm{dom}(A) \subset E \longrightarrow E$ is *invertible* on a normed vector space iff there is $B \in \mathcal{L}(E)$ with $AB = \mathrm{id}_E$ and $BA = \mathrm{id}_{\mathrm{dom}(A)}$. The *resovent set* $\rho(A)$ and the *spectrum* $\sigma(A)$ of A are then given by

$$\rho(A) := \big\{\lambda \in \mathbb{C} \,\big|\, \lambda - A \text{ is invertible}\big\}$$

and

$$\sigma(A) = \mathbb{C} \setminus \rho(A)\,,$$

respectively. The spectrum of an operator A is usually divided into *point spectrum* $\sigma_p(A)$, *continuous spectrum* $\sigma_c(A)$ and *residual spectrum* $\sigma_r(A)$ where

$$\sigma_p(A) = \big\{\lambda \in \sigma(A) \,\big|\, \mathit{ker}(\lambda - A) \neq \{0\}\big\}$$

$$\sigma_c(A) = \big\{\lambda \in \sigma(A) \,\big|\, (\lambda - A)^{-1} \text{ is densely defined but not bounded}\big\}$$

$$\sigma_r(A) = \big\{\lambda \in \sigma(A) \,\big|\, \mathit{range}(\lambda - A) \text{ is not dense in } E\big\}$$

Elements of $\sigma_p(A)$ are called *eigenvalues.*

1. Produce simple examples of operators showing that any of the spectral sets $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ can be non empty.

2. Let E be a normed vector space and $A \in \mathcal{L}(E)$ a compact operator. Show that $\sigma_p(A)$ is countable and the only possible accumulation point is $\lambda = 0$. Also $0 \in \sigma(A)$ if $\dim(E) = \infty$. It can be shown that any spectral value $\lambda \neq 0$ is an eigenvalue. [Hint: Show that for any given $\varepsilon > 0$ there are only finitely many eigenvalues with $|\lambda| \geq \varepsilon$.]

3. Let E be a normed vector space and $A \in \mathcal{L}(E)$ a compact operator. Show that

   $$\dim\big(\mathit{ker}\big[(\lambda - A)^n\big]\big) < \infty \text{ for any } \lambda \neq 0 \text{ and } n = 1, 2, \ldots$$

   [Use the fact that $\overline{\mathbb{B}}_E(0,1)$ is compact iff $\dim(E) < \infty$ and that eigenvectors to different eigenvalues are linearly independent.]

4. (Pohožaev's identity) Assume that $g \in \mathrm{C}(\mathbb{R}, \mathbb{R})$, $G(u) = \int_0^u g(v)\,dv$ and that $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary. Let u be a classical solution of

   $$\begin{cases} -\triangle u = g(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

   and show that it satisfies

   $$n \int_\Omega G(u)\,dx + \frac{2-n}{2} \int_\Omega u\, g(u)\,dx = \frac{1}{2} \int_{\partial\Omega} (\nabla u \cdot \nu)^2 (x \cdot \nu)\,d\sigma$$

[Hint: Use Gauss theorem with the vector field $V(x) = (x \cdot \nabla u)\nabla u$.]

5. Use Pohožaev's identity to prove that no nontrivial solution can exist for
$$\begin{cases} -\triangle u = |u|^p & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$
if $p > \frac{n+2}{n-2}$ and Ω is a star-shaped bounded Lipschitz domain in $\mathbb{R}^n$.

The Homework is due by May 3 2002.