Assignment 1

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1. Assume that $f \in \mathrm{C}\big([0,1]\big)$ and that

$$F \in \mathrm{C}\big([0,1]\times[0,1]\times\mathbb{R}\big)\,,\ \partial_u F \in \mathrm{C}\big([0,1]\times[0,1]\times\mathbb{R}\big)$$

and consider the integral equation

$$u(x) = \int_0^1 F\big(x,y,u(y)\big)\,dy + f(x)\,,\ 0 \le x \le 1\,.$$

Show that, if $\|\partial_u F\|_\infty < 1$, the integral equation has a unique solution $u \in \mathrm{C}\big([0,1]\big)$.

Let $\Omega \subset \mathbb{R}^n$ be open. A map $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is called *Carathéodory function* whenever
(i) $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}$.
(ii) $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \Omega$.

2. Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $p, q \ge 1$. Assume that

$$|f(x,s)| \le c|s|^{p/q} + g(x)$$

for some $g \in \mathrm{L}_q(\Omega)$. Prove that the Nemytzki operator (substitution operator) $N_f : \mathrm{L}_p(\Omega) \to \mathrm{L}_q(\Omega)$ defined through

$$\big(N_f u\big)(x) := f\big(x, u(x)\big)\,,\ x \in \Omega$$

is well-defined, continuous and maps bounded sets onto bounded sets.

3. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Assume that the Carathéodory function $f : \Omega\times\mathbb{R} \to \mathbb{R}$ satisfies

$$\underline{f} \le \frac{f(x,u) - f(x,v)}{u-v} \le \overline{f} \text{ and } f(\cdot,0) \in \mathrm{L}_2(\Omega)$$

with $\sigma\big(-\triangle_D\big) \cap [\underline{f}, \overline{f}] = \emptyset$. Show that

$$\begin{cases} \triangle u = f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

possesses a unique weak solution $u \in \mathring{\mathrm{H}}^1(\Omega)$.

4. (*Kolmogoroff*) Show that a subset $K \subset \mathrm{L}_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ is compact iff
   (i) K is closed and bounded.
   (ii) $\int_{|x|\geq N} |f(x)|^p \, dx \to 0\,,\ N \to \infty\,$, uniformly in $f \in K$.
   (iii) $\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, dx \to 0\,,\ |h| \to 0\,$, uniformly in $f \in K$.
   [Hint: Use the density of test functions in $\mathrm{L}_p(\mathbb{R}^n)$, the strong continuity of the translation semigroup on $\mathrm{L}_p(\mathbb{R}^n)$ and Arzéla-Ascoli.]

5. Let $1 \leq p < \infty$ and prove that

$$\Big(\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p \, dx\Big)^{1/p} \leq |h| \, \|u\|_{1,p}$$

   for $u \in \mathrm{W}^1_p(\mathbb{R}^n)$. Use this estimate and Kolmogoroff's characterization of compactness to show that $\mathrm{W}^1_p(\Omega) \hookrightarrow \mathrm{L}_p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open and bounded.

The Homework is due by April 19 2002