Assignment 2

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1. It can be shown that any compactly supported distribution can be viewed as some derivative of a continuous function. More precisely

   $\forall\, u \in \mathcal{E}' \;\exists\, f \in \mathrm{C}(\Omega)$ and $\alpha \in \mathbb{N}^n$ s.t.
   $$\langle u, \varphi \rangle = (-1)^{|\alpha|} \int_\Omega f(x) \partial^\alpha \varphi(x)\, dx \;\forall\, \varphi \in \mathcal{D}(\Omega)\,,$$
   if $\mathrm{supp}(u) \subset \Omega$. Let $\delta \in \mathcal{D}(\mathbb{R}^n)$ and find a simple representation of it as some derivative of a continuous function ($n \geq 2$).

2. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$ define $\tau_v u \in \mathcal{D}'(\mathbb{R}^n)$ through
   $$\langle \tau_v u, \varphi \rangle = \langle u, \tau_{-v} \varphi \rangle \;\forall\, \varphi \in \mathcal{D}(\mathbb{R}^n)$$
   where $\tau_v \varphi = \varphi(\cdot + v)$. Show that $\tau_v u$ is well-defined and prove that
   $$\frac{\tau_{he_j} u - u}{h} \longrightarrow \partial_j u \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ as } h \to 0$$
   for $j = 1, \ldots, n$.

3. Let $\varphi, \psi \in \mathcal{D}(\Omega)$ and $\alpha \in \mathbb{N}^n$. Determine the real numbers $c_{\alpha\beta}$, $\beta \leq \alpha$ for which the formula
   $$\partial^\alpha(\varphi\psi) = \sum_{\beta \leq \alpha} c_{\alpha,\beta}\, \partial^{\alpha-\beta}\varphi\, \partial^\beta \psi$$
   holds. By $\beta \leq \alpha$ it is meant that $\beta_j \leq \alpha_j$ for $j = 1, \ldots, n$.

4. Show that $u \in \mathcal{D}'\big((0,\infty)\big)$ given through
   $$\langle u, \varphi \rangle := \sum_{m \in \mathbb{N}} \big(\partial^m \varphi\big)\big(\frac{1}{m}\big)\,,\; \varphi \in \mathcal{D}\big((0,\infty)\big)$$
   is well-defined. Show that u has infinite order and that it cannot be extended to a distribution of the real line.

5. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial\Omega$ and denote by χ_Ω its characteristic function. Compute $\nabla \chi_\Omega$ in the sense of distributions and determine its support.

Homework due by Wednesday, October 24 2007