

Assignment 11

1. Let $A = [a_{jk}]_{1 \leq j, k \leq n} = A^T \geq \alpha > 0$, $b \in \mathbb{R}^n$, $p \in (1, \infty)$ and compute H^* for

$$H(x) = \frac{1}{p}|x|^p \text{ and } H(x) = \frac{1}{2} \sum_{j,k=1}^n a_{jk} x_j x_k + \sum_{i=1}^n b_i x_i.$$

2. Let $H : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. Its *subdifferential* at $p \in \mathbb{R}^n$ is the set given by

$$\partial H(p) = \{q \in \mathbb{R}^n \mid H(y) \geq H(p) + q \cdot (y - p), y \in \mathbb{R}^n\}.$$

Show that

$$q \in \partial H(p) \iff p \in \partial H^*(q) \iff p \cdot q = H(p) + H^*(q).$$

3. Let $H \in C^{-1}(\mathbb{R}^n)$ be convex and satisfy $\lim_{|p| \rightarrow \infty} H(p)/|p| = \infty$ and assume $g \in C^{-1}(\mathbb{R}^n)$. Prove that

$$\min_{y \in \mathbb{R}^n} \left\{ tH^*\left(\frac{x-y}{t}\right) + g(y) \right\} = \min_{y \in \mathbb{B}(x, Rt)} \left\{ tH^*\left(\frac{x-y}{t}\right) + g(y) \right\}$$

for $R = \|\nabla H(\nabla g)\|_\infty$.

4. Let $H \in C^{-1}(\mathbb{R}^n)$ be convex and satisfy $\lim_{|p| \rightarrow \infty} H(p)/|p| = \infty$ and assume $g_i \in C^{1-}(\mathbb{R}^n)$, $i = 1, 2$. Let u_i be a weak solution of

$$\begin{cases} u_t + H(\nabla u) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g_i & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Show the validity of the following contraction property

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_\infty \leq \|g_1 - g_2\|_\infty.$$

5. Compute the unique entropy solution of

$$\begin{cases} u_t + [\frac{u^2}{2}]_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

where g is given by

$$g(x) = \begin{cases} 1 & \text{if } x \in (-\infty, -1] \\ 0 & \text{if } x \in (-1, 0] \\ 2 & \text{if } x \in (0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

The Homework is due by Friday, April 11 2008