

Assignment 2

1. It can be shown that any distribution can be viewed as some derivative of a continuous function. More precisely

$$\forall u \in \mathcal{D}'(\Omega) \forall K = \overline{K} \subset \subset \Omega \exists f \in C(\Omega) \text{ and } \alpha \in \mathbb{N}^n \text{ s.t.}$$

$$\langle u, \varphi \rangle = (-1)^{|\alpha|} \int_{\Omega} f(x) \partial^{\alpha} \varphi(x) dx \quad \forall \varphi \in \mathcal{D}_K(\Omega).$$

For a proof see the book *Functional Analysis* by W. Rudin (p. 167). Let now $\delta \in \mathcal{D}'(\mathbb{R}^2)$ and find a simple representation of it as some derivative of a continuous function.

2. For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$ define $\tau_v u \in \mathcal{D}'(\mathbb{R}^n)$ through

$$\langle \tau_v u, \varphi \rangle = \langle u, \tau_{-v} \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$$

where $\tau_v \varphi = \varphi(\cdot - v)$. Show that $\tau_v u$ is well-defined and prove that

$$\frac{\tau_{-he_j} u - u}{h} \longrightarrow \partial_j u \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ as } h \rightarrow 0$$

for $j = 1, \dots, n$.

3. Let $\varphi, \psi \in \mathcal{D}(\Omega)$ and $\alpha \in \mathbb{N}^n$. Determine the real numbers $c_{\alpha\beta}$, $\beta \leq \alpha$ for which the formula

$$\partial^{\alpha}(\varphi\psi) = \sum_{\beta \leq \alpha} c_{\alpha\beta} \partial^{\alpha-\beta} \varphi \partial^{\beta} \psi$$

holds. By $\beta \leq \alpha$ it is meant that $\beta_j \leq \alpha_j$ for $j = 1, \dots, n$.

4. Show that $u \in \mathcal{D}'((0, \infty))$ given through

$$\langle u, \varphi \rangle := \sum_{m=1}^{\infty} (\partial^m \varphi)\left(\frac{1}{m}\right), \quad \varphi \in \mathcal{D}((0, \infty))$$

is well-defined. Show that u has infinite order and that it cannot be extended to a distribution of the real line.

5. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial\Omega$ and denote by χ_{Ω} its characteristic function. Compute $\nabla \chi_{\Omega}$ in the sense of distributions and determine its support.