

## Assignment 1

---

1. Show that  $v.p.\frac{1}{x}$  is a well-defined distribution. Recall that

$$\langle v.p.\frac{1}{x}, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} dx$$

for any test function  $\varphi \in \mathcal{D}(\mathbb{R})$ . Then prove that

$$u_{\varepsilon}^{\pm} \rightarrow \mp i\pi\delta + v.p.\frac{1}{x} \text{ as } \varepsilon \downarrow 0$$

for  $u_{\varepsilon}^{\pm}(x) := \frac{1}{x \pm i\varepsilon}$ ,  $x \in \mathbb{R}$ , in the sense of distributions.

2. Compute  $f'$  and  $f''$  for  $f(x) = \log|x|$ ,  $x \in \mathbb{R}$  and  $f(x) = |x|$ ,  $x \in \mathbb{R}$ , respectively, in the sense of distributions.

3. Let  $\rho : \mathbb{R}^n \mapsto \mathbb{R}$  be an integrable function with

$$\text{supp}(\rho) \subset \mathbb{B}(0, 1) \text{ and } \int_{\mathbb{R}^n} \rho(x) dx = 1.$$

Show that  $\rho_{\varepsilon} \rightarrow \delta$  in  $\mathcal{D}'(\mathbb{R}^n)$  as  $\varepsilon \rightarrow 0$  for

$$\rho_{\varepsilon}(x) := \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n.$$

4. Consider the function  $u$  defined as

$$u(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, (x, y) \in \mathbb{R} \times (0, \infty)$$

Then  $u \in C^{\infty}(\mathbb{R} \times (0, \infty))$ . Compute  $\lim_{y \rightarrow 0} u(\cdot, y)$ .

5. Given a continuous function  $f \in C(\Omega)$  with  $\text{supp}(f) \subset\subset \Omega$ , show that a sequence  $(\varphi_j)_{j \in \mathbb{N}}$  of test functions in  $\mathcal{D}(\Omega)$  can be found such that

$$\|\varphi_j - f\|_{\infty} := \sup_{x \in \Omega} |\varphi_j(x) - f(x)| \rightarrow 0 \text{ as } j \rightarrow \infty.$$