Assignment 8

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1. Let $\mathcal{A}$ be the general elliptic second order differential operator in divergence from on a bounded domain Ω with smooth boundary, that is,
$$\mathcal{A}u = -\nabla \cdot (A\nabla u) + (b|\nabla u) + cu$$
where the coefficents satisfy the assumptions of Chapter 8 of class. Let $\Phi \in \mathrm{Diff}^2(\Omega, \widetilde{\Omega})$, that is, Φ is invertible and
$$\Phi \in \mathrm{C}^2(\Omega, \widetilde{\Omega})\,,\ \Psi := \Phi^{-1} \in \mathrm{C}^2(\widetilde{\Omega}, \Omega)\,.$$
Letting $y := \Phi(x)$ and define $\tilde{u}(y) := u(\Psi(y))$, compute the operator $\tilde{\mathcal{A}}$ in the new variables, that is the operator satisfying
$$\widetilde{\mathcal{A}u} = \widetilde{\mathcal{A}}\tilde{u}\,.$$

2. Prove that the Neumann problem
$$\begin{cases} -\triangle u &= f \text{ in } \Omega\,, \\ \partial_\nu u &= 0 \text{ on } \partial\Omega \end{cases}$$
on a bounded domain with smooth boundary has a solution if and only if $\int_\Omega f\,dx = 0$.

3. Let $H(t, x)$ be the heat kernel (the fundamental solution for the heat equation introduced in class) and show that T defined through
$$\begin{cases} \big(T(t)u\big)(x) &:= \int_{\mathbb{R}^n} H(t, x-y)u(y)\,dy\,,\ x \in \mathbb{R}^n \\ T(0)u &:= u \end{cases}$$
is a C_0-semigroup of contractions on $\mathrm{L}_2(\mathbb{R}^n)$ but NOT on $\mathrm{L}_\infty(\mathbb{R}^n)$. A C_0-semigroup T on a Banach space E is called *analytic* if it allows for an analytic strongly continuous extension to a sector $\Sigma_\delta = [\arg(z) < \delta]$ of the complex plane for some $\delta \in (0, \pi/2]$, that is, if
   (i) $T(0) = \mathrm{id}_E\,,\ T(z_1 + z_2) = T(z_1)T(z_2)\,,\ z_1, z_2 \in \Sigma_\delta\,.$
   (ii) $T : \Sigma_\delta \to \mathcal{L}(E)$ is analytic.
   (iii) $\lim_{\Sigma_\delta \ni z \to 0} T(z)x = x$ for all $x \in E\,.$

   It can be shown that the above conditions are equivalent to
   (i) $T(t)E \subset \mathrm{dom}(A)\,,\ t > 0\,.$
   (ii) $\|tAT(t)\|_{\mathcal{L}(E)} \le c < \infty\,,\ t > 0\,.$

   where $-A : \mathrm{dom}(A) \subset E \longrightarrow E$ is the generator of T. Show that the C_0-semigroup of problem 1 is analytic.

4. Let $A : \mathrm{dom}(A) \subset E \longrightarrow E$ be defined through

$$
\begin{aligned}
&E = \mathrm{L}_2(0,1)\,, \\
&\mathrm{dom}(A) = \big\{ u \in \mathrm{H}^2(0,1) \,\big|\, u(0) = u(1) = 0 \big\}\,, \\
&Au = -\partial_{xx} u\,,\ u \in \mathrm{dom}(A)\,,
\end{aligned}
$$

and show that $-A$ generates an analytic C_0-semigroup on E.

5. For a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, for $b \in \mathrm{L}_\infty(\Omega)$ and $c \in \mathrm{L}_\infty(\Omega)$ let A be the operator induced by the Dirichlet form

$$
a(u,v) = \int_\Omega \big[(\nabla u \,|\, \nabla v) + (b \,|\, \nabla u) v + cuv \big]\, dx\,,\ u, v \in \mathring{\mathrm{H}}^1(\Omega)
$$

on $\mathrm{H}^{-1}(\Omega)$. Show that it generates a C_0-semigroup on $H^{-1}(\Omega)$.

Homework due by Friday, January 16 2015