

Assignment 4

1. Compute the Fourier transform with respect to the variable x of the following functions:

(i) $u_y(x) = e^{-y|x|}$, $y > 0$, $x \in \mathbb{R}$.

(ii) $u(x) = e^{-\frac{|x|^2}{2}}$, $x \in \mathbb{R}^n$.

(iii) $u(x) = \frac{y^2 - x^2}{(y^2 + x^2)^2}$, $y > 0$, $x \in \mathbb{R}$.

2. Show that G defined through $G(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$ for $x \in \mathbb{R}$ and $y > 0$ is harmonic, that is, $\Delta G = 0$, and conclude that

$$u_g(x, y) := \int_{-\infty}^{\infty} G(x - \tilde{x}, y) g(\tilde{x}) d\tilde{x}, \quad (x, y) \in \mathbb{R} \times (0, \infty)$$

represents a solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

for $g \in L_1(\mathbb{R})$. What is $\lim_{y \rightarrow \infty} u_g(\cdot, y)$?

3. Let $f \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(f) \subset \mathbb{B}(0, R)$ for $0 < R < \infty$. Show that its Fourier transform \hat{f} is holomorphic and satisfies

$$|\hat{f}(\xi + i\eta)| \leq c_N \frac{1}{(1 + |\xi|^2)^{N/2}} e^{R|\eta|}, \quad (\xi, \eta) \in \mathbb{R}^{2n}$$

for any $N \in \mathbb{N}$ and some constant c_N .

4. Assume $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ and let

$$T : \mathbb{R} \rightarrow \mathcal{S}(\mathbb{R}^n), \quad t \rightarrow \varphi(\cdot - ta).$$

Prove that $T \in C^1(\mathbb{R}, \mathcal{S}(\mathbb{R}^n))$ and compute

$$\dot{T}(0) \in \mathcal{L}(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)) \hat{=} \mathcal{S}(\mathbb{R}^n).$$

5. Let $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and consider the homogeneous heat equation

$$\begin{cases} u_t - \Delta u = 0, & \text{in } (0, \infty) \times \mathbb{R}^n \\ u(0) = u_0, & \text{in } \mathbb{R}^n \end{cases}$$

Prove that it has a unique solution

$$u \in C^\infty([0, \infty), \mathcal{S}(\mathbb{R}^n))$$

and derive a representation formula for it.