Assignment 10

Let $\Omega \subset \mathbb{R}^n$ be open. A map $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is called *Carathéodory function* whenever
(i) $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}$.
(ii) $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \Omega$.

1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $p, q \geq 1$. Assume that
$$|f(x,s)| \leq c|s|^{p/q} + g(x)$$
for some $g \in \mathrm{L}_q(\Omega)$. Prove that the Nemytzki operator (substitution operator) $N_f : \mathrm{L}_p(\Omega) \to \mathrm{L}_q(\Omega)$ defined through
$$\big(N_f u\big)(x) := f\big(x, u(x)\big)\,,\ x \in \Omega$$
is well-defined, continuous and maps bounded sets onto bounded sets.

2. Let H be a Hilbert space. Prove that
$$x_n \to x\,,\ y_n \rightharpoonup y\ (n \to \infty) \Rightarrow (x_n|y_n) \to (x|y)\ (n \to \infty)\,.$$

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. Prove that the equation
$$\begin{cases} -\triangle u = e^{-u} & \text{in } \Omega\,, \\ \quad u = 0 & \text{on } \partial\Omega\,, \end{cases}$$
possesses a solution.

4. (*Kolmogoroff*) Show that a subset $K \subset \mathrm{L}_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ is compact iff
(i) K is closed and bounded.
(ii) $\int_{|x|\geq N} |f(x)|^p\,dx \to 0\,,\ N \to \infty\,$, uniformly in $f \in K$.
(iii) $\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p\,dx \to 0\,,\ |h| \to 0\,$, uniformly in $f \in K$.
[Hint: Use the density of test functions in $\mathrm{L}_p(\mathbb{R}^n)$, the strong continuity of the translation semigroup on $\mathrm{L}_p(\mathbb{R}^n)$ and Arzéla-Ascoli.]

5. Let $1 \leq p < \infty$ and prove that
$$\Big(\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p\,dx\Big)^{1/p} \leq |h|\,\|u\|_{1,p}$$
for $u \in \mathrm{W}^1_p(\mathbb{R}^n)$. Use this estimate and Kolmogoroff's characterization of compactness to show that $\mathrm{W}^1_p(\Omega) \hookrightarrow \mathrm{L}_p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open and bounded.

Homework due by Wednesday, February 18 2015.