

Assignment 9

1. Let $H(t, x)$ be the heat kernel (the fundamental solution for the heat equation introduced in class) and show that T defined through

$$\begin{cases} (T(t)u)(x) & := \int_{\mathbb{R}^n} H(t, x-y)u(y) dy, \quad x \in \mathbb{R}^n \\ T(0)u & := u \end{cases}$$

is a C_0 -semigroup of contractions on $L_2(\mathbb{R}^n)$ but NOT on $L_\infty(\mathbb{R}^n)$.

A C_0 -semigroup T on a Banach space E is called *analytic* if it allows for an analytic strongly continuous extension to a sector $\Sigma_\delta = [\arg(z) < \delta]$ of the complex plane for some $\delta \in (0, \pi/2]$, that is, if

- (i) $T(0) = \text{id}_E$, $T(z_1 + z_2) = T(z_1)T(z_2)$, $z_1, z_2 \in \Sigma_\delta$.
- (ii) $T : \Sigma_\delta \rightarrow \mathcal{L}(E)$ is analytic.
- (iii) $\lim_{\Sigma_\delta \ni z \rightarrow 0} T(z)x = x$ for all $x \in E$.

It can be shown that the above conditions are equivalent to

- (i) $T(t)E \subset \text{dom}(A)$, $t > 0$.
- (ii) $\|tAT(t)\|_{\mathcal{L}(E)} \leq c < \infty$, $t > 0$.

where $-A : \text{dom}(A) \subset E \rightarrow E$ is the generator of T . Show that the C_0 -semigroup of problem 1 is analytic.

2. Let $-A : \text{dom}(A) \subset E \rightarrow E$ be the generator of an analytic C_0 -semigroup T on E . Let $f \in C^\rho([0, T], E)$ for some $\rho \in (0, 1)$ and show that the mild solution $u : [0, T] \rightarrow E$ of

$$\dot{u} + Au = f(t), \quad u(0) = x \in E,$$

given by

$$u(t) = T(t)x + \int_0^t T(t-\tau)f(\tau) d\tau, \quad t \in [0, T]$$

is actually differentiable for $t > 0$.

3. Let $A : \text{dom}(A) \subset E \rightarrow E$ be defined through

$$\begin{aligned} E &= L_2(0, 1), \\ \text{dom}(A) &= \{u \in H^2(0, 1) \mid u(0) = u(1) = 0\}, \\ Au &= -\partial_{xx}u, \quad u \in \text{dom}(A), \end{aligned}$$

and show that $-A$ generates an analytic C_0 -semigroup on E .

4. Define $T(t) \in \mathcal{L}(L_2(\mathbb{R}^n))$ through

$$\begin{cases} (1 - \Delta)^{-t} = \mathcal{F}^{-1}(1 + |\xi|^2)^{-t} \mathcal{F}, & t > 0 \\ \text{id}_{L_2(\mathbb{R}^n)}, & t = 0. \end{cases}$$

Show that T is a C_0 -semigroup on $L_2(\mathbb{R}^n)$. What is its generator?

5. For a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, for $b \in L_\infty(\Omega)$ and $c \in L_\infty(\Omega)$ let A be the operator induced by the Dirichlet form

$$a(u, v) = \int_{\Omega} [(\nabla u | \nabla v) + (b | \nabla u)v + cuv] dx, \quad u, v \in \mathring{H}^1(\Omega)$$

on $H^{-1}(\Omega)$. Show that it generates a C_0 -semigroup on $H^{-1}(\Omega)$.