

Assignment 8

1. Let \mathcal{A} be the general elliptic second order differential operator in divergence form on a bounded domain Ω with smooth boundary, that is,

$$\mathcal{A}u = -\nabla \cdot (A\nabla u) + (b|\nabla u) + cu$$

where the coefficients satisfy the assumptions of Chapter 8 of class. Let $\Phi \in \text{Diff}^2(\Omega, \tilde{\Omega})$, that is, Φ is invertible and

$$\Phi \in C^2(\Omega, \tilde{\Omega}), \Psi := \Phi^{-1} \in C^2(\tilde{\Omega}, \Omega).$$

Letting $y := \Phi(x)$ and define $\tilde{u}(y) := u(\Psi(y))$, compute the operator $\tilde{\mathcal{A}}$ in the new variables, that is the operator satisfying

$$\tilde{\mathcal{A}}u = \tilde{\mathcal{A}}\tilde{u}.$$

2. Prove that the Neumann problem

$$\begin{cases} -\Delta u &= f \text{ in } \Omega, \\ \partial_\nu u &= 0 \text{ on } \partial\Omega \end{cases}$$

on a bounded domain with smooth boundary has a solution if and only if $\int_\Omega f \, dx = 0$.

3. For $\alpha \in (0, 1)$, $0 \leq s_0 < s_1$ and $s = (1 - \alpha)s_0 + \alpha s_1$ prove the interpolation inequality

$$\|u\|_{H^s} \leq c \|u\|_{H^{s_0}}^{1-\alpha} \|u\|_{H^{s_1}}^\alpha, \quad u \in H^{s_1}$$

first for \mathbb{R}^n and then for a bounded domain with smooth boundary.

4. Let $f : \Omega \rightarrow \mathbb{R}$ be measurable. Define

$$\mu_f(t) := |\{x \in \Omega : |f(x)| > t\}|$$

Let $p > 0$ and assume $f \in L_p(\Omega)$. Prove that

$$\mu_f(t) \leq t^{-p} \|f\|_p^p$$

and that

$$\|f\|_p^p = p \int_0^\infty t^{p-1} \mu_f(t) \, dt.$$

5. Let $1 \leq q < r < \infty$ and $T : L_q(\Omega) \cap L_r(\Omega) \rightarrow L_q(\Omega) \cap L_r(\Omega)$ be a linear operator such that

$$\mu_{Tf}(t) \leq (T_1 \|f\|_q / t)^q \text{ and } \mu_{Tf}(t) \leq (T_2 \|f\|_r / t)^r$$

for some constants T_1 and T_2 . Then T can be extended to an operator $T \in \mathcal{L}(L_p(\Omega))$ for any $p \in (q, r)$ and

$$\|Tf\|_p \leq c T_1^\alpha T_2^{1-\alpha} \|f\|_p, \quad f \in L_q(\Omega) \cap L_r(\Omega)$$

where $1/p = (1 - \alpha)/r + \alpha/q$.

[Hint: For $s > 0$ use $f = f\chi_{\{|f|>s\}} + f\chi_{\{|f|\leq s\}} = f_1 + f_2$ to prove that

$$\mu_{Tf}(t) \leq \mu_{Tf_1}(t/2) + \mu_{Tf_2}(t/2)$$

and then use the previous problem and Fubini's theorem.]