

Assignment 16

1. (Pohožaev's identity) Assume that $g \in C(\mathbb{R}, \mathbb{R})$, $G(u) = \int_0^u g(v) dv$ and that $\Omega \subset \mathbb{R}^n$ is bounded with smooth boundary. Let u be a classical solution of

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and show that it satisfies

$$n \int_{\Omega} G(u) dx + \frac{2-n}{2} \int_{\Omega} u g(u) dx = \frac{1}{2} \int_{\partial\Omega} (\nabla u \cdot \nu)^2 (x \cdot \nu) d\sigma$$

[Hint: Use Gauss theorem with the vector field $V(x) = (x \cdot \nabla u) \nabla u$.]

2. Use Pohožaev's identity to prove that no nontrivial solution can exist for

$$\begin{cases} -\Delta u = |u|^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if $p > \frac{n+2}{n-2}$ and Ω is a star-shaped bounded Lipschitz domain in \mathbb{R}^n .

3. Let X be a normed vector space. Prove that a convex functional $\phi : X \rightarrow \mathbb{R}$ is continuous at $x \in X$ if it is bounded in a neighborhood of x . Give an example of a convex functional which is nowhere continuous.
4. Let $\beta \in C^\infty(\mathbb{R})$ satisfying $\beta'(\mathbb{R}) \subset [\delta, \sigma]$ for $\delta, \sigma \in (0, \infty)$. Give a weak formulation of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_\nu u + \beta(u) = 0 & \text{on } \Omega. \end{cases}$$

in an open bounded domain $\Omega \subset \mathbb{R}^n$ and prove that it possesses a weak solution.

5. For an open and bounded $\Omega \subset \mathbb{R}^n$ let

$$\mathcal{A} = \{u \in H_0^1(\Omega, \mathbb{R}^m) \mid u = g \text{ on } \partial\Omega, |u| = 1 \text{ a.e.}\}.$$

Show that ϕ defined by $\phi(u) = \frac{1}{2} \int_{\Omega} |Du(x)|^2 dx$ has at least one minimizer in \mathcal{A} (if $\mathcal{A} \neq \emptyset$) and that any minimizer satisfies

$$\int_{\Omega} Du(x) : Dv(x) dx = \int_{\Omega} |Du(x)|^2 u(x)v(x) dx, \\ v \in H_0^1(\Omega, \mathbb{R}^m) \cap L_\infty(\Omega, \mathbb{R}^m).$$