Assignment 11

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1. Let $0 \neq \alpha \in \mathbb{R}$ and consider the initial value problem for Euler's equation

$$\sum_{k=1}^{n} x_k \partial_{x_k} u = \alpha u \, , \; u(x_1, \ldots, x_{n-1}, 1) = g(x_1, \ldots, x_{n-1}) \, .$$

Show that its solution satisfies the functional equation

$$u(\lambda x) = \lambda^{\alpha} u(x) \, , \; x \neq 0 \, , \; \lambda > 0 \, .$$

What is the behavior of the solution at $x = 0$?

2. Consider the quasilinear initial value problem

$$\begin{cases} u_t + u\, u_x = 0 \, , & (t, x) \in (0, \infty) \times \mathbb{R} \\ u(0, x) = g(x) \, , & x \in \mathbb{R} \end{cases}$$

Solve the equation and analyze the possible onset of singularities. What conditions on g would prevent the solution from developing singularities?

3. Let $u \in \mathrm{C}^1\big(\mathbb{B}(0,1)\big)$ be a solution of

$$a(x, y) u_x + b(x, y) u_y = -u$$

and assume that $a(x, y)x + b(x, y)y > 0$ for $(x, y) \in \mathbb{S}^1$. Show that $u \equiv 0$ then.

4. Consider the equation

$$\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0 \, .$$

Any function u with

$$\int_{\mathbb{R}^2} \big[R(u) \phi_y + S(u) \phi_x \big] \, d(x, y) = 0 \, , \; \phi \in \mathrm{C}_0^{\infty}(\mathbb{R}^2)$$

is called *weak solution*. Assume that u is continuously differentiable away from some curve parametrized by $\big(s(y), y\big) \, , y \in \mathbb{R}$, across which it has a jump discontinuity. Conclude that

$$s'(y) = \frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)}$$

where $u^{\pm}$ indicate the one sided limits of u approaching the curve.

5. Consider the eikonal equation

$$c^2(u_x^2 + u_y^2) = 1$$

Let γ_t be the level line $[u(x, y) = t]$ of a solution u. Show that a point (x, y) moves in a direction perpendicular to γ_t at constant speed c (along a characteristic).

The Homework is due by Wednesday, February 20 2013