Assignment 7

1. For $\alpha \in (0,1)$, $0 \leq s_0 < s_1$ and $s = (1-\alpha)s_0 + \alpha s_1$ prove the interpolation inequality
$$\|u\|_{\mathrm{H}^s} \leq c\,\|u\|_{\mathrm{H}^{s_0}}^{1-\alpha}\|u\|_{\mathrm{H}^{s_1}}^{\alpha}\,,\ u \in \mathrm{H}^{s_1}$$
first for $\mathbb{R}^n$ and then for a bounded domain with smooth boundary.

2. Let Banach spaces E_j, $j = 0, 1, 2$, be given with
$$E_2 \overset{d}{\hookrightarrow} E_1 \hookrightarrow E_0\,.$$
Show that, given $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that
$$\|u\|_{E_1} \leq \varepsilon\|u\|_{E_2} + c_\varepsilon\|u\|_{E_0}\,,\ u \in E_2\,.$$

3. Show that the trace operator $\gamma_{\mathbb{H}^n}$ satisfies
$$\gamma_{\mathbb{H}^n}\big(\mathrm{H}^2(\mathbb{R}^n)\big) = \mathrm{H}^{3/2}(\partial\mathbb{H}^n)\,.$$

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $\alpha > 0$ and $f \in \mathrm{L}_2(\Omega)$. Find the boundary value problem for which
$$\int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega uv\,dx + \alpha\int_{\partial\Omega} u\,v\,d\sigma_{\partial\Omega} = \int_\omega fv\,dx\;\forall v \in \mathrm{H}^1(\Omega)$$
is the appropriate weak formulation and show that it possesses a unique solution.

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and find the natural weak formulation of the bvp
$$\triangle^2 u = f \in \mathrm{L}_2(\Omega)\,,\ u = 0\,,\ \partial_\nu u = 0 \text{ on } \partial\Omega$$
and prove that it has a unique solution.

Homework due by Friday, December 4 2009