

## Assignment 5

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1. Compute a fundamental solution  $G_2$  for the wave operator  $\partial_t^2 - \Delta$  on  $\mathbb{R} \times \mathbb{R}^2$ . [Hint: Let  $G_3$  be the fundamental solution for the wave operator on  $\mathbb{R} \times \mathbb{R}^3$  introduced in class and show that

$$\langle G_3(t, x, x_3), \varphi(t, x) \mathbf{1}(x_3) \rangle$$

can be made sense of for  $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$ . Define  $G_2$  through

$$\langle G_2, \varphi \rangle = \langle G_3, \varphi \mathbf{1}(x_3) \rangle = \frac{1}{4\pi} \int_0^\infty \frac{1}{t} \int_{\mathbb{S}_t^2} \varphi(t, x) d\sigma_{\mathbb{S}_t^2}(x) dt$$

and use the fact that  $\varphi$  is independent of  $x_3$  to simplify the integral.]

2. A fundamental solution  $G(\cdot, y)$  for the Laplacian  $-\Delta$  on a domain  $\Omega \subset \mathbb{R}^n$  with pole at  $x = y$  and satisfying  $G(\cdot, y)|_{\partial\Omega} = 0$  is called *Green's function* for the Dirichlet problem in  $\Omega$  (considered as a function of  $(x, y) \in \Omega \times \Omega$ ). Compute a Green's function for the Dirichlet problem in the half-space  $\mathbb{H}^n = \mathbb{R}^{n-1} \times (0, \infty)$ ,  $n \geq 2$ .
3. Let  $G$  be the fundamental solution for the heat operator  $\partial_t - \Delta$  on  $\mathbb{R} \times \mathbb{R}^n$  introduced in class. Show that

$$G(t, \cdot) \rightarrow \delta, \quad t \rightarrow 0+, \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and find a representation for the solution of the Cauchy problem

$$\partial_t u - \Delta u = f \in L_{1,loc}([0, \infty), \mathcal{S}'), \quad u(0, \cdot) = u_0 \in \mathcal{S}'$$

on  $(0, \infty) \times \mathbb{R}^n$  in terms of  $G$ .

4. Compute the general solution to the following initial value problem for the wave equation

$$\partial_t^2 u - \partial_x^2 u = f(t, x), \quad u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1,$$

in  $(0, \infty) \times \mathbb{R}$  in terms of a fundamental solution for the wave operator. What is the regularity of the solution if

$$(f, u_0, u_1) \in C([0, \infty), \mathcal{S}) \times \mathcal{S} \times \mathcal{S}?$$

5. Let  $u : \Omega \rightarrow \mathbb{R}$  be a harmonic function on the domain  $\Omega$ . Prove *Gauss's law of the arithmetic mean*

$$u(x) = \frac{1}{\omega_n r^{n-1}} \int_{|x-y|=r} u(y) d\sigma_r(y)$$

valid for all  $x$  and  $r$  such that  $\mathbb{B}(x, r) \subset \Omega$ .

[Hint: Use Green's formula with  $u$  and a Green's function for the Dirichlet problem on  $\mathbb{B}(x, r)$ .]

**Homework due by Friday, November 13 2009**