Assignment 4

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1. Compute the Fourier transform with respect to the variable x of the following functions:
   (i) $u_y(x) = e^{-y|x|}\,,\ y > 0\,,\ x \in \mathbb{R}$.
   (ii) $u(x) = e^{-\frac{|x|^2}{2}}\,,\ x \in \mathbb{R}^n\,.$
   (iii) $u(x) = \frac{y^2-x^2}{(y^2+x^2)^2}\,,\ y > 0\,,\ x \in \mathbb{R}$.

2. Show that G defined through $G(x, y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$ for $x \in \mathbb{R}$ and $y > 0$ is harmonic, that is, $\triangle G = 0$, and conclude that
$$u_g(x, y) := \int_{-\infty}^{\infty} G(x - \tilde{x}, y)g(\tilde{x})\,d\tilde{x}\,,\ (x, y) \in \mathbb{R} \times (0, \infty)$$
   represents a solution of
$$\begin{cases} \triangle u &= 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u &= g \text{ on } \mathbb{R} \times \{0\} \end{cases}$$
   for $g \in \mathrm{L}_1(\mathbb{R})$. What is $\lim_{y\to\infty} u_g(\cdot, y)$?

3. Let $f \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subset \mathbb{B}(0, R)$ for $0 < R < \infty$. Show that its Fourier transform $\hat{f}$ is holomorphic and satisfies
$$|\hat{f}(\xi + i\eta)| \le c_N \frac{1}{(1 + |\xi|^2)^{N/2}} e^{R|\eta|}\,,\ (\xi, \eta) \in \mathbb{R}^{2n}\,.$$

4. Assume $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $a \in \mathbb{R}^n$ and let
$$T : \mathbb{R} \to \mathcal{S}(\mathbb{R}^n)\,,\ t \to \varphi(\cdot - t\,a)\,.$$
   Prove that $T \in \mathrm{C}^1\big(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)\big)$ and compute
$$\dot{T}(0) \in \mathcal{L}\big(\mathbb{R}, \mathcal{S}(\mathbb{R}^n)\big) \,\hat{=}\, \mathcal{S}(\mathbb{R}^n)\,.$$

5. Let $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and consider the homogeneous heat equation
$$\begin{cases} u_t - \triangle u = 0\,, & \text{in } (0, \infty) \times \mathbb{R}^n \\ u(0) = u_0\,, & \text{in } \mathbb{R}^n \end{cases}$$
   Prove that it has a unique solution
$$u \in \mathrm{C}^\infty\big([0, \infty), \mathcal{S}(\mathbb{R}^n)\big)$$
   and derive a representation formula for it.

Homework due by Wednesday, November 2 2009