Assignment 16

1. Assume that u is a viscosity solution of

(HJ) $$u_t + H(\nabla u, x) = 0 \text{ in } \mathbb{R}^n \times (0,\infty)\,.$$

Show that $-u$ is a viscosity solution of

$$v_t - H(-\nabla v, x) = 0 \text{ in } \mathbb{R}^n \times (0,\infty)\,.$$

2. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of viscosity solutions of (HJ) and suppose that $u_k \to u$ uniformly as $k \to \infty$. Show that u is also a solution of (HJ) in the viscosity sense.

3. Let u^ε be a classical solution of the parabolic equation

$$u_t + H(\nabla u, x) - \varepsilon \sum_{i,j=1}^{n} a^{ij}(x)\partial_{ij}u^\varepsilon \text{ in } \mathbb{R}^n \times (0,\infty)\,,$$

where the coefficients are smooth and satisfy

$$\underline{\alpha}|\xi|^2 \le \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \le \bar{\alpha}|\xi|^2\,,$$

for some $0 < \underline{\alpha} \le \bar{\alpha} < \infty$. Assume that H is continuous and that u^ε converges uniformly to a function u. Prove that u is a viscosity solution of (HJ).

4. Let u^i be solutions of (HJ) with initial datum g^i for $i = 1, 2$. Assume that H satisfies

$$|H(p,x) - H(q,x)| \le C|p-q|$$
$$\text{and } |H(p,x) - H(p,y)| \le C|x-y|\big(1+|p|\big)$$

for $x, y, p, q \in \mathbb{R}^n$ and some $C \ge 0$. Show that

$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_\infty \le \|g^1 - g^2\|_\infty\,,\ t \ge 0\,.$$

5. Show that $1 - |x|$ is a viscosity solution of $|u'| = 1$ on $(-1,1)$ with boundary conditions $u(\pm 1) = 0$. This means that $|v'(x_0)| \le 1[\ge 1]$ whenever $u - v$ has a maximum [minimum] at $x_0 \in (-1,1)$ and $v \in \mathrm{C}^\infty(-1,1)$. Show that $|x| - 1$ is NOT a viscosity solution of $|u'| = 1$ but one of $-|u'| = -1$ in $(-1,1)$ with the same boundary conditions. How do you explain the fact that the two equations have different solutions?

Homework due on Friday, May 7 2010