Assignment 15

Let $\Omega \subset \mathbb{R}^n$ be open. A map $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is called *Carathéodory function* whenever
(i) $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}$.
(ii) $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \Omega$.

1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $p, q \geq 1$. Assume that
$$|f(x,s)| \leq c|s|^{p/q} + g(x)$$
for some $g \in \mathrm{L}_q(\Omega)$. Prove that the Nemytzki operator (substitution operator) $N_f : \mathrm{L}_p(\Omega) \to \mathrm{L}_q(\Omega)$ defined through
$$\big(N_f u\big)(x) := f\big(x, u(x)\big)\,,\ x \in \Omega$$
is well-defined, continuous and maps bounded sets onto bounded sets.

2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Assume that the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies
$$\underline{f} \leq \frac{f(x,u) - f(x,v)}{u - v} \leq \overline{f} \text{ and } f(\cdot, 0) \in \mathrm{L}_2(\Omega)$$
with $\sigma\big(-\triangle_D\big) \cap [\underline{f}, \overline{f}] = \emptyset$. Show that
$$\begin{cases} \triangle u = f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
possesses a unique weak solution $u \in \overset{\circ}{\mathrm{H}}{}^1(\Omega)$.

3. Let H be a Hilbert space. Prove that
$$x_n \to x\,,\ y_n \rightharpoonup y\ (n \to \infty) \Rightarrow (x_n|y_n) \to (x|y)\ (n \to \infty)\,.$$

4. Let E be a normed vector space and $A \in \mathcal{L}(E)$ a compact operator. Show that
$$\dim(ker\big[(\lambda - A)^n\big]) < \infty \text{ for any } \lambda \neq 0 \text{ and } n = 1, 2, \ldots$$
[Use the fact that $\overline{\mathbb{B}}_E(0,1)$ is compact iff $\dim(E) < \infty$ and that eigenvectors to different eigenvalues are linearly independent.]

5. Let $K \in \mathcal{L}(H)$ be a compact operator. Show that it maps weakly convergent sequences to convergent sequences.

Homework due on Wednesday, April 21 2010