Assignment 14

1. Assume that $f \in \mathrm{C}\big([0,1]\big)$ and that
$$F \in \mathrm{C}\big([0,1]\times[0,1]\times\mathbb{R}\big)\,,\ \partial_u F \in \mathrm{C}\big([0,1]\times[0,1]\times\mathbb{R}\big)$$
and consider the integral equation
$$u(x) = \int_0^1 F\big(x,y,u(y)\big)\,dy + f(x)\,,\ 0 \le x \le 1\,.$$
Show that, if $\|\partial_u F\|_\infty < 1$, the integral equation has a unique solution $u \in \mathrm{C}\big([0,1]\big)$.

2. Let $f \in \mathrm{C}\big(\mathbb{B}(0,1),\mathbb{R}^n\big)$ be such that
$$|f(x)| \le 1 \text{ for } |x| = 1\,.$$
Show that f has a fixed point in $\mathbb{B}(0,1)$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with smooth boundary. Prove that the equation
$$\begin{cases} -\triangle u = e^{-u} & \text{in } \Omega\,,\\ \quad u = 0 & \text{on } \partial\Omega\,,\end{cases}$$
possesses a solution.

4. (*Kolmogoroff*) Show that a subset $K \subset \mathrm{L}_p(\mathbb{R}^n)$ $(1 \le p < \infty)$ is compact iff
 (i) K is closed and bounded.
 (ii) $\int_{|x|\ge N} |f(x)|^p\,dx \to 0\,,\ N \to \infty\,,$ uniformly in $f \in K$.
 (iii) $\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p\,dx \to 0\,,\ |h| \to 0\,,$ uniformly in $f \in K$.
 [Hint: Use the density of test functions in $\mathrm{L}_p(\mathbb{R}^n)$, the strong continuity of the translation semigroup on $\mathrm{L}_p(\mathbb{R}^n)$ and Arzéla-Ascoli.]

5. Let $1 \le p < \infty$ and prove that
$$\Big(\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p\,dx\Big)^{1/p} \le |h|\,\|u\|_{1,p}$$
for $u \in \mathrm{W}^1_p(\mathbb{R}^n)$. Use this estimate and Kolmogoroff's characterization of compactness to show that $\mathrm{W}^1_p(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}_p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ open and bounded.

Homework due by Friday, April 9 2010.