# Assignment 12

---

1. Let $A = [a_{jk}]_{1\leq j,k\leq n} = A^T \geq \alpha > 0$, $b \in \mathbb{R}^n$, $p \in (1,\infty)$ and compute $H^*$ for
$$H(x) = \frac{1}{p}|x|^p \text{ and } H(x) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}x_jx_k + \sum_{i=1}^{n} b_ix_i\,.$$
2. Let $H : \mathbb{R}^n \to \mathbb{R}$ be convex. Its *subdifferential* at $p \in \mathbb{R}^n$ is the set given by
$$\partial H(p) = \big\{ q \in \mathbb{R}^n \,\big|\, H(y) \geq H(p) + q\cdot(y-p)\,,\ y \in \mathbb{R}^n \big\}\,.$$
Show that
$$q \in \partial H(p) \iff p \in \partial H^*(q) \iff p\cdot q = H(p) + H^*(q)\,.$$
3. Let $H \in \mathrm{C}^{-1}(\mathbb{R}^n)$ be convex and satisfy $\lim_{|p|\to\infty} H(p)/|p| = \infty$ and assume $g \in \mathrm{C}^{-1}(\mathbb{R}^n)$. Prove that
$$\min_{y\in\mathbb{R}^n} \big\{ tH^*\big(\frac{x-y}{t}\big) + g(y)\big\} = \min_{y\in\mathbb{B}(x,Rt)} \big\{ tH^*\big(\frac{x-y}{t}\big) + g(y)\big\}$$
for $R = \|\nabla H(\nabla g)\|_\infty$.
4. Let $H \in \mathrm{C}^{-1}(\mathbb{R}^n)$ be convex and satisfy $\lim_{|p|\to\infty} H(p)/|p| = \infty$ and assume $g_i \in \mathrm{C}^{1-}(\mathbb{R}^n)$, $i = 1,2$. Let $u_i$ be a weak solution of
$$\begin{cases} u_t + H(\nabla u) = 0 & \text{in } \mathbb{R}^n \times (0,\infty)\,,\\ u = g_i & \text{on } \mathbb{R}^n \times \{0\}\,.\end{cases}$$
Show the validity of the following contraction property
$$\|u_1(\cdot,t) - u_2(\cdot,t)\|_\infty \leq \|g_1 - g_2\|_\infty\,.$$
5. Compute the unique entropy solution of
$$\begin{cases} u_t + [\frac{u^2}{2}]_x = 0 & \text{in } \mathbb{R} \times (0,\infty)\\ u = g & \text{on } \mathbb{R} \times \{0\}\end{cases}$$
where $g$ is given by
$$g(x) = \begin{cases} 1 & \text{if } x \in (-\infty,-1]\\ 0 & \text{if } x \in (-1,0]\\ 2 & \text{if } x \in (0,1]\\ 0 & \text{otherwise.}\end{cases}$$

The Homework is due by Friday, March 5 2010