Assignment 11

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1. Let $0 \neq \alpha \in \mathbb{R}$ and consider the initial value problem for Euler's equation
$$\sum_{k=1}^{n} x_k \partial_{x_k} u = \alpha u\,,\ u(x_1,\dots,x_{n-1},1) = g(x_1,\dots,x_{n-1})\,.$$
Show that its solution satisfies the functional equation
$$u(\lambda x) = \lambda^\alpha u(x)\,,\ x \neq 0\,,\ \lambda > 0\,.$$
What is the behavior of the solution at $x = 0$?

2. Consider the quasilinear initial value problem
$$\begin{cases} u_t + u\,u_x = 0\,, & (t,x) \in (0,\infty)\times\mathbb{R} \\ u(0,x) = g(x)\,, & x \in \mathbb{R} \end{cases}$$
Solve the equation and analyze the possible onset of singularities. What conditions on g would prevent the solution from developing singularities?

3. Let $u \in \mathrm{C}^1\big(\mathbb{B}(0,1)\big)$ be a solution of
$$a(x,y)u_x + b(x,y)u_y = -u$$
and assume that $a(x,y)x + b(x,y)y > 0$ for $(x,y) \in \mathbb{S}^1$. Show that $u \equiv 0$ then.

4. Consider the equation
$$\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0\,.$$
Any function u with
$$\int_{\mathbb{R}^2} \big[R(u)\phi_y + S(u)\phi_x\big]\,d(x,y) = 0\,,\ \phi \in \mathrm{C}_0^\infty(\mathbb{R}^2)$$
is called *weak solution*. Assume that u is continuously differentiable away from some curve parametrized by $(s(y),y)\,,\ y \in \mathbb{R}$ across which it has a jump discontinuity. Conclude that
$$s'(y) = \frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)}$$
where $u^\pm$ indicate the one sided limits of u approaching the curve.

5. Consider the eikonal equation

$$c^2(u_x^2 + u_y^2) = 1$$

Let γ_t be the level line $[u(x, y) = t]$ of a solution u. Show that a point (x, y) moves in a direction perpendicular to γ_t at constant speed c (along a characteristic).

The Homework is due by Monday, March 1 2010