Math 205 Spring Term 2006

Midterm Examination

Print your name: ________________ ________________

Print your ID #: ________________________________

You have 50 minutes to solve the problems. Good luck!

1. Let M be a complete metric space and $f : M \to M$ a contraction. Denoting by x_0 the fixed point of f, prove that

$$d(x, x_0) \leq \frac{1}{1-r} d(x, f(x))$$

for any $x \in M$ where $r \in (0, 1)$ is the Lipschitz constant of f.

2. Let $f, g \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R})$ be positive functions. Show that $fg \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R})$ and compute $D(fg)$. Show that, if fg attains a minimum at x, then $\nabla f(x)$ and $\nabla g(x)$ are linearly dependent.

3. Let $f \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R})$ and assume that

$$x \in L := f^{-1}(5) := \{y \in \mathbb{R}^n \mid f(y) = 5\}\,.$$

If $\gamma \in \mathrm{C}^1\big((0,1), L\big)$ is a curve through x, show that $\nabla f(x)$ is orthogonal to the curve γ at x.

4. Let $f \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R})$ with $D^2 f(x) > 0$ for some $x \in \mathbb{R}^n$. Show that, in a neighborhood of x, the graph

$$G_f := \{(x, f(x)) \mid x \in \mathbb{R}^n\}$$

of f lies above its tangent plane at x.

5. Let $f \in \mathrm{C}([a,b],\mathbb{R})$ and $g \in \mathrm{C}([c,d],\mathbb{R})$. Show that

$$\int_R f(x)g(y)\,d(x,y) = [\int_a^b f(x)\,dx][\int_c^d g(x)\,dx]$$

for $R := [a,b] \times [c,d]$.