# Assignment 21

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1. Consider $n-$dimensional *polar coordinates* as given by

$$\begin{aligned} x_1 &= r\cos(\varphi_1) \\ x_2 &= r\sin(\varphi_1)\cos(\varphi_2) \\ x_3 &= r\sin(\varphi_1)\sin(\varphi_2)\cos(\varphi_3) \\ &\vdots \\ x_{n-1} &= r\sin(\varphi_1)\ldots\sin(\varphi_{n-2})\cos(\varphi_{n-1}) \\ x_n &= r\sin(\varphi_1)\ldots\sin(\varphi_{n-2})\sin(\varphi_{n-2}) \end{aligned}$$

for

$$(r,\varphi_1,\ldots,\varphi_{n-2},\varphi_{n-1}) \in \\ D := (0,\infty)\times(0,\pi)\times\cdots\times(0,\pi)\times(-\pi,\pi)\,.$$

Show that the change of variables map $g$ defined by

$$(r,\varphi_1,\ldots,\varphi_{n-2},\varphi_{n-1}) \mapsto (x_1,\ldots,x_n), D\to\mathbb{R}^n$$

is one-to-one and onto $\mathbb{R}^n\setminus\{0\}$. Prove that

$$\det\big[Dg(r,\varphi_1,\ldots,\varphi_{n-2},\varphi_{n-1})\big] \\ = r^{n-1}[\sin(\varphi_1)]^{n-2}[\sin(\varphi_2)]^{n-3}\cdots\sin(\varphi_{n-2})\,.$$

2. Let $f\in \mathrm{C}\big(\overline{\mathbb{B}}_{\mathbb{R}^n}(0,1),\mathbb{R}\big)$ satisfy

$$f(x)=g(|x|_2)$$

for some $g\in\mathrm{C}\big([0,1],\mathbb{R}\big)$. Compute the integral of $f$ over the unit ball $\mathbb{B}_{\mathbb{R}^n}(0,1)$.

3. Compute the integral of the function $f$ defined by

$$f(x,y)=\begin{cases} x+y\,, & (x,y)\in[0,1]\times[0,1] \text{ s.t. } x^2\le y\le 2x^2 \\ 0\,, & \text{otherwise}\,. \end{cases}$$

4. Determine the volume of the region between the sphere given by the solution set of $x^2+y^2+z^2=8$ and the paraboloid given by $4z=x^2+y^2+4$. [Hint: Use cylindrical coordinates.]

5. Let $g : D \to \mathbb{R}^n$ be one-to-one for $D \overset{o}{\subset} \mathbb{R}^n$. Assume that
$$f \in \mathrm{C}^1(\overline{D}, \mathbb{R}^n)\,.$$
Let $\{C_j \,|\, j = 1, \ldots, N\}$ be a collection of disjoint hypercubes of side $\delta > 0$ fully contained in $D$ stemming from a uniform partition of a parallelopiped containing $D$. Show that
$$\sum_{j=1}^{N} \mathrm{vol}[g(C_j)] \to \mathrm{vol}[g(D)] \;\text{ as } \delta \to 0\,.$$

Homework due by Thursday, April 11 2006.