Assignment 19

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1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by
$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}\,, & (x,y) \neq (0,0) \\ 0\,, & (x,y) = (0,0) \end{cases}$$
and compute its mixed second derivatives.

2. Denote $\mathbb{R}^n \backslash \{0\}$ by $\dot{\mathbb{R}}^n$. A function $f : \dot{\mathbb{R}}^n \to \mathbb{R}$ is called homogeneous of degree k if
$$f(tx) = t^k f(x)\,,\ t > 0\,,\ x \in \dot{\mathbb{R}}^n\,.$$
Show that
$$\nabla f(x) \cdot x = k f(x)$$
if f is differentiable.

3. Let $f \in \mathrm{C}^1(D, \mathbb{R})$ for some convex $D \overset{o}{\subset} \mathbb{R}^n$ and $x, y \in D$. Show that there exists $\xi \in D$ such that
$$f(y) - f(x) = \nabla f(\xi) \cdot (y - x)\,.$$
Why is convexity of D needed?

4. Classify the critical points of
$$f : \mathbb{R}^2 \to \mathbb{R}\,,\ (x,y) \mapsto x^2 + y^2 - x^4 - y^4\,.$$

5. Let $f \in \mathrm{C}^3(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$ be such that
$$f(x) = 0\,,\ \nabla f(x) = 0\,,\ D^2 f(x) = 0\,.$$
Assuming that there is $\alpha \in \mathbb{N}^n$ with $|\alpha| = 3$ and $\partial^\alpha f(x) \neq 0$, what can you say about the critical point x?

Homework due by Thursday, April 27 2006